Resistive Diode Network Theory

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Circuit theory can be a wonderful synthesis of logic and intuition. Most of the theorems of circuit analysis were discovered by intuitive thinking — and then proved analytically. The analytical investigation proves the validity of the intuitive approach and in addition, points out limitations or conditions under which a method can be expected to work.

In this article, we (that means you, and us), will analytically demonstrate the truth of some theorems concerned with networks containing only resistors, diodes, and sources, hereafter known as RDS networks. In order to exaggerate the elegance of the analytical method we will begin building our collection of theorems from scratch, presupposing little or no knowledge of circuit theory on the part of the reader.

We will begin by considering the constraints placed on the voltages and currents in a network by Kirchoff's laws.

1. The sum of the voltages across any sequence of branches forming a closed loop is zero.
2. The sum of the currents leaving any point (node) is zero.

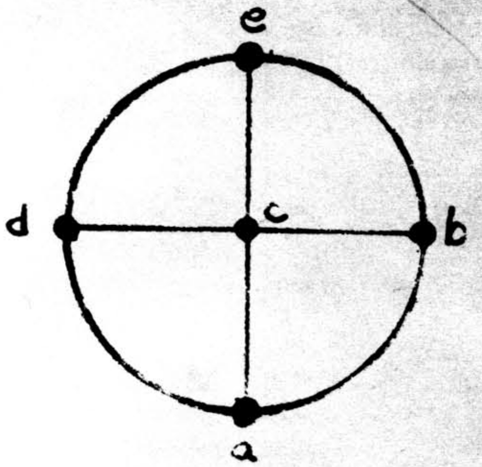


Figure 1

Consider figure 1, a network graph. We will represent the voltage on branch **be** as E_{be}, considered positive if it would tend to drive positive current through a resistor from **b** to **e.** Clearly, $E_{be} = -E_{eb}$. Similarly we represent the current along branch **eb** as I_{be} if positive current flow is from **b** to **e.** (Current flow is opposite to electron flow. Credit

Ben Franklin for any additional confusion caused.) Again, of course, $I_{be} = {}^{-}I_{eb}$. As examples of the restrictions imposed on our network by Kirchoff's laws, we observe:

1. $E_{ab}+E_{bc}+E_{ce}+E_{ed}+E_{da} = 0$
2. $I_{be}+I_{bc}+I_{ba} = 0$

To each node of our network let us assign a number N(i). Let N(a) = 0. To find the number assigned to any other node, consider some sequence of branches forming a path from that node to node **a.** Now add the voltages along the chosen path and assign the total to the node.

Ex. Consider node **e.** We might pick path **ed, dc, ca.**

Then $N(e) = E_{ed}+E_{dc}+E_{ca}$.

Theorem: The voltage across any branch is the difference between the numbers assigned to the nodes at the ends of the branch.

Proof: Consider the voltage across branch **be,** E_{be}. N(e) is the sum of the voltages across the branches in a path to **a.** Call this path *e.* Similarly, N(b) is the sum of the voltages along path *be* to node *a.* Now branch *be* together with the branches in *b* and those in *e* forms a closed loop. Thus by Kirchoff's voltage law $E_{be}+N(e) - N(b) = 0$. So $E_{be} = N(b) - N(e)$.

Theorem: In any network with current and voltage distribution satisfying Kirchoff's laws, the product of the voltage across a branch with the current through the branch, summed over all branches is zero.

Proof: Consider again branch **be.** $E_{be}I_{be}$ is equal to $(N(b) - N(e))I_{be} = N(b)I_{be}+N(e)I_{eb}$. (Note carefully subscripts, changed minus signs, etc.) if we sum this expression over all branches, we see that we will get many terms containing N(b). In fact we will have a term of the form $N(b)I_{bi}$ for each branch leaving node **b.** Then the sum of the terms containing N(b) will be just N(b) times the total of the currents leaving node **b.** But by Kirchoff's current law this is zero. Of course the total of those terms containing N(e) will be zero by this same reasoning. Thus $\sum_{\text{over all branches}} EI = 0$.

Physically we have just demonstrated conservation of power. Isn't logic (and Kirchoff) wonderful. With this one tool we will be able to prove several more clever theorems.

Let's try our hand at showing a uniqueness theorem for RDS networks. The existence of a uniqueness theorem means that given a set of voltages and currents in a network that satisfy Kirchoff's laws, that it is the only set which does so. The method of proof is not straightforward, as we will first assume there are two solutions, then show that they are identical.

In order to go any further, we have to inquire a bit about the current-voltage relationships imposed by a resistor or diode.

Figure 2

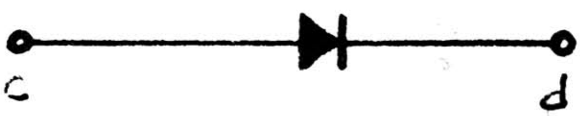


Figure 3

A resistor forces E to be related to I by Ohm's law. $E_{ab} = RI_{ab}$. In the case of a diode we have the constraints:

$$I_{cd} > 0 \text{ and } E_{cd} = 0 \text{ when } I_{cd} > 0$$
$$E_{cd} < 0 \text{ and } I_{cd} = 0 \text{ when } E_{cd} < 0$$

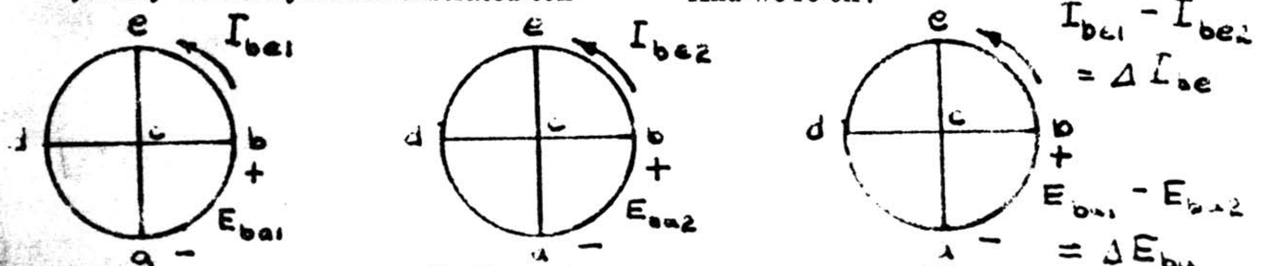
From the above constraints we see that EI for a diode is always zero. This is because either E or I is always zero.

With this background we can demonstrate that if we have two identical RDS networks (same value resistors and sources) and a voltage and current distribution on each that satisfies both

1. Kirchoff's laws
2. Ohm's law and the diode constraints

then the current through any given resistor is the same in both distributions.

And we're off:



distribution 1 distribution 2 difference distribution

Figure 4

Figure 4 shows graphically two distributions on the same network graph. Let these be the two solutions for our graph. Then make a difference distribution on the same network graph by assigning to each branch in the network a current equal to the difference of the currents in the same branch in each of the solutions. In other words, we assign to branch **ij** the current equal to the current through **ij** in distribution 1 less the current through **ij** in distribution 2. Figure 4 may help clarify this a bit. Similarly, assign to each branch the voltage equal to the difference between that branch's voltage in each of the two distributions.

Lemma: The voltage and currents on the difference distribution satisfy Kirchoff's laws.

Proof: We will consider the voltage law. Let E_{ij1} be the voltage across branch **ij** in distribution 1. Let E_{ij2} be the voltage across this branch in distribution 2. Since Kirchoff's laws must be satisfied around any loop in both distributions 1 and 2,

$$\sum_{\text{branches around a loop}} E_{ij1} = 0$$

$$\sum_{\text{branches around same loop}} E_{ij2} = 0$$

$$\sum_{\text{branches around same loop}} (E_{ij1} - E_{ij2}) = 0$$

Similarly we can show Kirchoff's current law is satisfied on the difference distribution.

Lemma: The difference voltage and current satisfy Ohm's law across any branch of the network that contains a resistor.

Proof: $E_{ij1} = I_{ij1}R_{ij}$ and $E_{ij2} = I_{ij2} R_{ij}$.

Thus $E_{ij1} - E_{ij2} = R_{ij}(I_{ij1} - I_{ij2})$.

Lemma: Any branch containing a voltage source has a difference voltage of zero.

Proof: Think hard and wave hands in air. If this doesn't work, try to remember definition of a voltage source.

Lemma: Any branch containing a current source has a difference current of zero.

Proof: Same as above.

Lemma: Across any diode, the difference voltage times the difference current is greater than or equal to zero.

Proof: (Not obvious) Let E_1 and I_1 be the current and voltage on the diode branch for distribution 1, and E_2 and I_2 the current and voltage on the same branch for distribution 2.

$$\Delta E \Delta I = E_1I_1 + E_2I_2 - (E_1I_2 + E_2I_1)$$

$$\Delta E \Delta I = (E_1 - E_2)(I_1 - I_2)$$

By a preceding lemma, $E_1I_1 = E_2I_2 = 0$

Now, I_1 and I_2 are greater than or equal to zero. E_1 and E_2 are less than or equal to zero. Thus both I_1E_2 and I_2E_1 are less than or equal to zero.

Hence:

$$\Delta E \Delta I = -(I_1E_2 + I_2E_1) = 0$$

In fact $\Delta I \Delta E$ can equal zero only if the diode has not changed state. This is apparent upon examination of above expression for $\Delta I \Delta E$.

Finally we have accumulated sufficient artillery to handle a uniqueness theorem.

Assume the existence of two distributions satisfying Kirchoff's laws and the element constraints on our network. Since the ΔE's and ΔI's satisfy Kirchoff's laws on the difference network we have:

$$\sum_{\text{over all branches}} \Delta I \Delta E = 0$$

We have seen that for branches containing a source, either ΔI or ΔE is zero. Thus the contribution to $\sum(\Delta I \Delta E)$ from the source branches is zero. Consider now the contribution from a resistive branch.

$$\Delta I \Delta E = \Delta I \text{ x } R\Delta I = (\Delta I)^2 R$$

since Ohm's law is satisfied on the difference network.

Thus:

$$\sum (\Delta I)^2 R + \sum_{\text{diode branches}} \Delta I \Delta E = 0$$

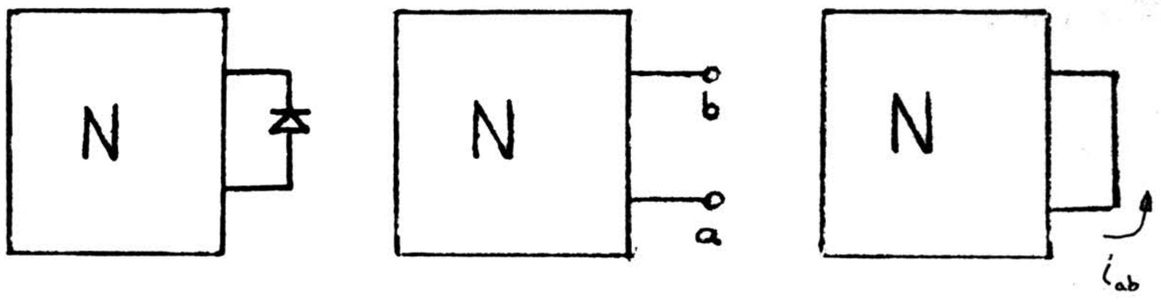
But $(\Delta I)^2$ is greater than or equal to zero, and $\Delta I \Delta E$ across a diode branch is greater than or equal to zero. Thus the above sum can equal zero only if each of its terms is zero. That is, each ΔI through a resistor equals zero, and each $\Delta I \Delta E$ across a diode equals zero.

Hence, (at last) given two distributions on a network satisfying Kirchoff's laws and the element constraints

1. The current through a given resistor is the same in both distributions.
2. The diodes are in the same state in both distributions.

This is what we set out to prove. It follows immediately that if the resistors, current sources, and open diodes contain a link set or if the voltage sources, resistors, and shorted diodes contain a tree, then the two distributions are exactly alike. (That last comment was for people familiar with circuit theory vocabulary. If you aren't one of these people, you probably won't notice the loop hole left if the comment wasn't made.)

Theorem: Most RDS networks containing only one diode have solutions.



a b c
Figure 5

work of only resistors and sources. Let us consider only those networks where there does exist a solution to the network obtained by throwing away the diode, and where there also exists a solution to the network obtained by shorting the diode. It is easily shown that these networks are at least those where the voltage sources do not contain a tie set with the diode shorted and where the current sources do not contain a cut set with the diode opened. (More higher plane, but unnecessary vocabulary.)

What are the implications of the nonexistence of a solution? Obviously E_{ab} in figure 5b is greater than zero. If not we could let the diode be open and this would be the solution. And I_{ab} is less than zero in figure 5c. Forming a difference network as before we get a difference distribution on the network graph such that

1. The voltage on any branch that contained a voltage source is zero.
2. The current through any branch that contained a current source is zero.
3. The difference current and voltage satisfy Ohm's law on those branches that contained resistors.
4. The difference voltage and difference current on the diode branch are as shown in figure 6.

But we know $\sum_{\text{all branches}}(\Delta E \Delta I) = 0$ and on all the resistors $\Delta E \Delta I \geqq 0$. Hence $\sum_{\text{all branches}}(\Delta E \Delta I)$ can be zero only if $\Delta E \Delta I$ on a diode branch is zero or negative. A glance at figure 6 shows that $\Delta E \Delta I$ on the diode branch is $-E_{ab}I_{ab}$. But if $E_{ab}>0$ and $I_{ab}<0$, $-E_{ab}I_{ab}>0$. Hence the assumption of nonexistence of a solution leads to a contradiction and therefore must be wrong.

We have thus demonstrated that given an RDS network with only one diode such that the resistor-source networks obtained by shorting the diode branch and by opening the diode branch each have a solution, there exists a solution of the RDS network.

Obviously, this is not general enough. Let us assume every N-diode RDS network has a solution. Consider an (N+1)-diode network such as that shown in figure 7a. We shall assume the N diode net has solutions both with the external diode branch opened and shorted. Assuming the nonexistence of a solution and forming the difference distribution as before we see that the assumption is contradictory.

Again $\Delta E \Delta I = 0$. Again $\Delta E \Delta I$ on a resist-

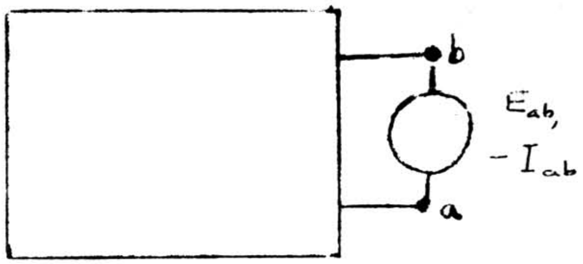
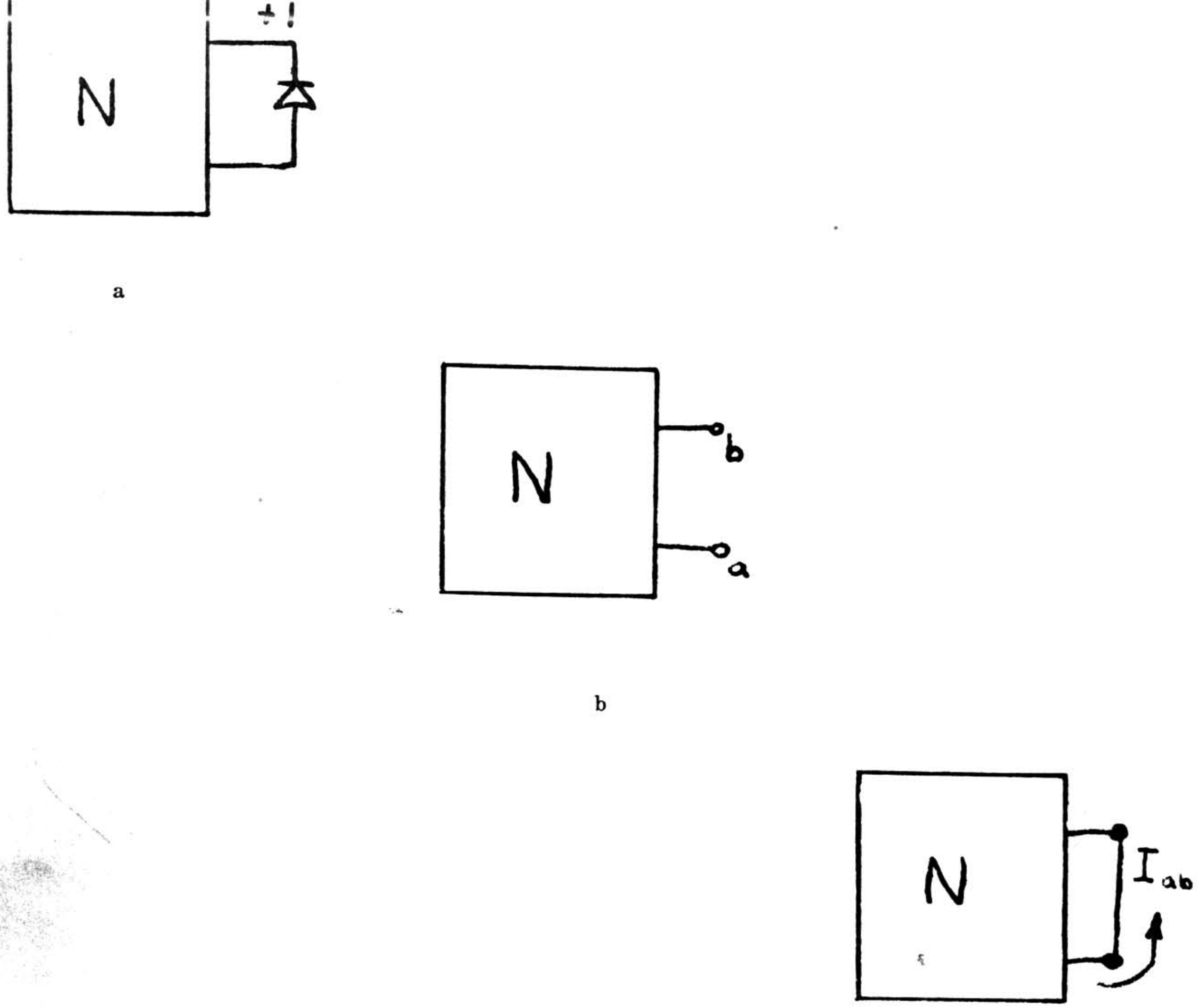


Figure 6



a

b

c

Figure 7

or branch is greater than or equal to zero. Since the diode constraints are satisfied within the N diode network in both (b) and (c) of figure 7, $\Delta E \Delta I$ on these diodes is greater than or equal to zero. This implies that $\Delta E \Delta I$ on the external diode branch is less than or equal to zero which conflicts with our assumption that there was no solution to the (N+1)-diode network. Thus this assumption is wrong. In other words, given an N-diode network connected to a diode such that there exists a solution to the N-diode network, both with the external diode branch opened and with the external diode branch closed, there exists a solution to the (N+1)-diode network.

Now we have arrived. A bit of concentration convinces us that if we have an RDS network such that the resistor-source network formed by opening some diode branches and shorting others has a solution for all possible combinations of openings and shortings; then the RDS network has a solution.

At last we have accomplished what we set out to achieve. We know that almost all RDS networks

1. Have a solution.
2. Have at most one solution.

Certainly it was fun demonstrating this. In addition we have set our analysis of diode circuits on a sound basis. We can attack a problem knowing we aren't wasting our time — there is some solution. And if by chance we hit upon a solution we can stop looking — there isn't going to be another one. Satisfying, what?