12.803. Change of Coordinates (non–orthogonal)

General coordinate change

There is a fairly straightforward mathematical procedure for changing coordinates from one system to another, even if the second is not orthogonal. Suppose we have a function $\psi(\mathbf{x})$ and wish to express it and its derivatives as functions of the new coordinates $\boldsymbol{\xi}$. We could use the chain rule to find

$$\frac{\partial \psi}{\partial x_i} = \frac{\partial \xi_j}{\partial x_i}\frac{\partial \psi}{\partial \xi_j} \tag{1}$$

But this may not be adequate, for the following reason. We wish to have coefficients in the final equations expressed as functions of the new coordinates; however, quantities such as

$$\frac{\partial \xi_1}{\partial x_3}$$

are more likely to be known as functions of $\mathbf{x}$.

To accomplish the goal of having all terms expressed in the new coordinates, we begin with the opposite form

$$\frac{\partial \psi}{\partial \xi_i} = \frac{\partial x_j}{\partial \xi_i}\frac{\partial \psi}{\partial x_j} \quad or \quad \nabla_x \psi = \mathbf{T}\nabla_\xi \psi \tag{2}$$

and assume that the $\frac{\partial x_j}{\partial \xi_i}$ terms are functions of $\boldsymbol{\xi}$. We can express derivatives in the old coordinate system in terms of derivates in the new system by inverting the transformation matrix:

$$\frac{\partial \psi}{\partial x_i} = \left[\frac{\partial x_i}{\partial \xi_j}\right]^{-1}\frac{\partial \psi}{\partial \xi_j} \quad or \quad \nabla_\xi \psi = \mathbf{T}^{-1}\nabla_x \psi \tag{3}$$

In terms of the Jacobian matrix

$$\frac{\partial(A,B,C)}{\partial(\xi_1,\xi_2,\xi_3)} \equiv det \begin{pmatrix} \frac{\partial A}{\partial \xi_1} & \frac{\partial A}{\partial \xi_2} & \frac{\partial A}{\partial \xi_3} \\ \frac{\partial B}{\partial \xi_1} & \frac{\partial B}{\partial \xi_2} & \frac{\partial B}{\partial \xi_3} \\ \frac{\partial C}{\partial \xi_1} & \frac{\partial C}{\partial \xi_2} & \frac{\partial C}{\partial \xi_3} \end{pmatrix}$$

we have

$$\frac{\partial \psi}{\partial x_1} = \frac{\partial(\psi,x_2,x_3)}{\partial(\xi_1,\xi_2,\xi_3)}\Big/\frac{\partial(x_1,x_2,x_3)}{\partial(\xi_1,\xi_2,\xi_3)}$$

etc.

Example

If we take polar coordinates as a specific case, we have the relationship between the old and new coordinates

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z'$$

So that the transformation matrix matrix $T_{ij} = \frac{\partial x_j}{\partial \xi_i}$ in (2) is

$$\mathbf{T} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The inverse is

$$\mathbf{T}^{-1} = \begin{pmatrix} \cos\theta & -\frac{1}{r}\sin\theta & 0 \\ \sin\theta & \frac{1}{r}\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

so that

$$\psi_x = \cos\theta\,\psi_r - \frac{1}{r}\sin\theta\,\psi_\theta$$
$$\psi_y = \sin\theta\,\psi_r + \frac{1}{r}\cos\theta\,\psi_\theta$$
$$\psi_z = \psi_{z'}$$

as obtained before (but the previous derivation used the orthogonality).

Change in vertical coordinate

If we switch from $x,\ y,\ z$ to $x',\ y',\ \xi$, the transformation matrix is

$$\mathbf{T} = \begin{pmatrix} 1 & 0 & \dfrac{\partial z}{\partial x'} \\ 0 & 1 & \dfrac{\partial z}{\partial y'} \\ 0 & 0 & \dfrac{\partial z}{\partial \xi} \end{pmatrix}$$

and its inverse is

$$\mathbf{T}^{-1} = \begin{pmatrix} 1 & 0 & -\dfrac{\partial z}{\partial x'}\Big/\dfrac{\partial z}{\partial \xi} \\ 0 & 1 & -\dfrac{\partial z}{\partial y'}\Big/\dfrac{\partial z}{\partial \xi} \\ 0 & 0 & 1\Big/\dfrac{\partial z}{\partial \xi} \end{pmatrix}$$

Thus we can replace horizontal gradients

$$\nabla \longrightarrow \nabla - \frac{\nabla z}{z_\xi}\frac{\partial}{\partial \xi}$$

vertical derivatives

$$\frac{\partial}{\partial z} \longrightarrow \frac{1}{z_\xi}\frac{\partial}{\partial \xi}$$

and time derivatives

$$\frac{\partial}{\partial t} \longrightarrow \frac{\partial}{\partial t} - \frac{z_t}{z_\xi}\frac{\partial}{\partial \xi}$$

in our original equations.

First, we note that the material derivative becomes

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla + \frac{1}{z_\xi}(w - z_t - \mathbf{u}\cdot\nabla z)\frac{\partial}{\partial \xi}$$

and we can define the “vertical” velocity ω as

$$\omega = \frac{1}{z_\xi}(w - z_t - \mathbf{u}\cdot\nabla z)$$

so that the material derivative becomes

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla + \omega\frac{\partial}{\partial \xi}$$

With this definition, we note that $w = \frac{D}{Dt}z$ as we might expect.

Transformed equations

The horizontal momentum equations become

$$\frac{D}{Dt}\mathbf{u} + f\hat{\mathbf{k}} \times \mathbf{u} = -\frac{1}{\rho}\nabla p - \nabla\varphi \qquad (e.1)$$

with $\varphi = gz$ being the geopotential; the hydrostatic balance is

$$\frac{\partial}{\partial\xi}\varphi = -\frac{1}{\rho}\frac{\partial}{\partial\xi}p \qquad (e.2a)$$

while the conservation of mass gives

$$\frac{1}{\rho}\frac{D}{Dt}\rho + \nabla\cdot\mathbf{u} - \frac{1}{z_\xi}\mathbf{u}_\xi\cdot\nabla z + \frac{1}{z_\xi}\frac{\partial}{\partial\xi}(\frac{D}{Dt}z) = 0$$

implying

$$\frac{1}{\rho}\frac{D}{Dt}\rho + \frac{1}{z_\xi}\frac{D}{Dt}z_\xi + \nabla\cdot\mathbf{u} + \frac{\partial}{\partial\xi}\omega$$

or

$$\frac{1}{p_\xi}\frac{D}{Dt}p_\xi + \nabla\cdot\mathbf{u} + \frac{\partial}{\partial\xi}\omega = 0 \qquad (e.3)$$

Finally, the thermodynamic equation becomes

$$\frac{D}{Dt}\rho - \frac{1}{c_s^2}\frac{D}{Dt}p = 0 \qquad (e.4a)$$

in general. The potential vorticity (with η being the entropy) is

$$q = -\frac{g}{p_\xi}(\nabla_3\times\mathbf{u} + f\hat{\mathbf{k}})\cdot\nabla_3\eta \qquad (e.5)$$

with the ∇_3 notation indicating the vertical derivatives are included.

Vertical coordinate function of pressure

When the vertical coordinate is a function of pressure $\xi = \xi(p)$ or $p = p(\xi)$, we can define $p_\xi \equiv -g\rho_c(\xi)$ and simplify the equations to

$$\frac{D}{Dt}\mathbf{u} + f\hat{\mathbf{k}} \times \mathbf{u} = -\nabla\varphi \qquad (p.1)$$

$$\frac{\partial}{\partial\xi}\varphi = g\frac{\rho_c}{\rho} \equiv b \qquad (p.2)$$

$$\nabla \cdot \mathbf{u} + \frac{1}{\rho_c}\frac{\partial}{\partial\xi}(\rho_c\omega) = 0 \qquad (p.3)$$

$$\frac{D}{Dt}\rho + \omega\frac{g\rho_c}{c_s^2} = 0 \quad or \quad \frac{D}{Dt}b + \omega\left[-g\frac{\rho_{c\xi}}{\rho} - \frac{g^2\rho_c^2}{\rho^2c_s^2}\right] = 0$$

The last equation can also be written

$$\frac{\partial}{\partial t}b + \mathbf{u}\cdot\nabla b + \omega\left[-\frac{g\rho_c\rho_\xi}{\rho^2} - \frac{g^2\rho_c^2}{\rho^2c_s^2}\right] = 0$$

or

$$\frac{\partial}{\partial t}b + \mathbf{u}\cdot\nabla b + \omega\mathcal{S} = 0 \qquad (p.4)$$

with the stratification parameter $\mathcal{S}$

$$\mathcal{S} \equiv \frac{\rho_c^2}{\rho^2}N^2 = b_\xi - b\frac{\rho_{c\xi}}{\rho_c} - \frac{b^2}{c_s^2} \qquad (p.5a)$$

defined in terms of the Brunt-Väisälä frequency

$$N^2 = -g\frac{1}{\rho}\frac{\partial}{\partial z}\rho - \frac{g^2}{c_s^2} = -g\frac{\rho_\xi}{\rho_c} - \frac{g^2}{c_s^2} = \frac{g^2}{b^2}b_\xi - \frac{g^2}{b}\frac{\rho_{c\xi}}{\rho_c} - \frac{g^2}{c_s^2} \qquad (p.5b)$$

The PV is

$$q = \frac{1}{\rho_c}(\nabla_3 \times \mathbf{u} + f\hat{\mathbf{k}})\cdot\nabla_3\eta \qquad (p.6)$$

Thermodynamics

For an ideal gas, we can simplify the thermodynamics using $\eta = c_p \ln\theta$

$$\frac{D}{Dt}\theta = 0 \qquad (p.7)$$

with the potential temperature being

$$\theta = \theta_0\frac{\rho_0}{\rho}\left(\frac{p}{p_0}\right)^{1/\gamma}$$

Thus, the buoyancy becomes

$$b = g\frac{\rho_c}{\rho_0}\left(\frac{p}{p_0}\right)^{-1/\gamma}\frac{\theta}{\theta_0} \equiv G(\xi)\theta \qquad (p.8)$$

With a little work, you can substitute (p.8) into (p.4), using $c_s^2 = \gamma p/\rho$ to show that (p.7) holds. The Brunt-Väisälä frequency is

$$N^2 = g\frac{\partial}{\partial z}\ln\theta = g\frac{\rho}{\rho_c}\frac{\partial}{\partial\xi}\ln\theta \quad , \quad \mathcal{S} = g\frac{\rho_c}{\rho}\frac{\partial}{\partial\xi}\ln\theta$$