12.803
Quasi-Balanced Motions in the Ocean and atmosphere

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Lecture II

Barotropic dynamics

Two-dimensional, constant density fluid

$$\frac{Du}{Dt} - fv = -\frac{\partial \phi}{\partial x} + F^{(x)}$$
$$\frac{Dv}{Dt} + fu = -\frac{\partial \phi}{\partial y} + F^{(y)}$$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \, ,$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} \, .$$

Streamfunction $u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x} \, ,$

Vorticity $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \nabla^2 \psi$

Barotropic circulation theorem

$$\mathcal{C} = \oint \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{r}$$

$$\begin{aligned}
\frac{D\mathcal{C}}{Dt} &= \frac{D}{Dt}\oint \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{r} \\
&= \oint \frac{D\boldsymbol{u}}{Dt} \cdot \mathrm{d}\boldsymbol{r} + \oint \boldsymbol{u} \cdot \frac{D\mathrm{d}\boldsymbol{r}}{Dt} \\
&= \oint \left[-f\boldsymbol{z} \times \boldsymbol{u} - \nabla\phi + \boldsymbol{F}\right] \cdot \mathrm{d}\boldsymbol{r} + \oint \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{u} \\
&= \oint \left[-f\boldsymbol{z} \times (\boldsymbol{z} \times \nabla\psi) + \boldsymbol{F}\right] \cdot \mathrm{d}\boldsymbol{r} \\
&= \oint \left[f\nabla\psi + \boldsymbol{F}\right] \cdot \mathrm{d}\boldsymbol{r} \\
&= \oint \left[-\psi\nabla f + \boldsymbol{F}\right] \cdot \mathrm{d}\boldsymbol{r}
\end{aligned}$$

Baroclinic circulation theorem: Kelvin's theorem

$$
\begin{aligned}
\frac{D\mathcal{C}}{Dt} &= \frac{D}{Dt}\oint \boldsymbol{u}\cdot \mathrm{d}\boldsymbol{r} \\
&= \oint \frac{D\boldsymbol{u}}{Dt}\cdot \mathrm{d}\boldsymbol{r} + \oint \boldsymbol{u}\cdot\frac{D\mathrm{d}\boldsymbol{r}}{Dt} \\
&= \oint \left[-f\boldsymbol{z}\times\boldsymbol{u} - \frac{1}{\rho}\nabla p + \boldsymbol{F}\right]\cdot \mathrm{d}\boldsymbol{r} \\
&= \oint \left[-f\boldsymbol{z}\times\boldsymbol{u} \boxed{- \frac{\nabla\rho\times\nabla p}{\rho^2}} + \boldsymbol{F}\right]\cdot \mathrm{d}\boldsymbol{r}
\end{aligned}
$$

Vorticity equation

$$\zeta = \hat{\boldsymbol{z}} \cdot \nabla \times \boldsymbol{u}$$

$$\hat{\boldsymbol{z}} \cdot \nabla \times \left\{ \begin{array}{l}
\dfrac{\partial \mathbf{u}}{\partial t} \longrightarrow -\dfrac{\partial^2 u}{\partial y \partial t} + \dfrac{\partial^2 v}{\partial x \partial t} = \dfrac{\partial \zeta}{\partial t} \ ; \\[2em]
(\mathbf{u} \cdot \nabla)\mathbf{u} \longrightarrow -\dfrac{\partial}{\partial y}\left[u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y}\right] + \dfrac{\partial}{\partial x}\left[u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y}\right] \\
\qquad = u\left(\dfrac{\partial^2 v}{\partial x^2} - \dfrac{\partial^2 u}{\partial y \partial x}\right) + v\left(\dfrac{\partial^2 v}{\partial y \partial x} - \dfrac{\partial^2 u}{\partial y^2}\right) \\
\qquad\quad -\dfrac{\partial u}{\partial y}\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y}\right) + \dfrac{\partial v}{\partial x}\left(\dfrac{\partial u}{pdx} + \dfrac{\partial v}{\partial y}\right) \\
\qquad = \mathbf{u} \cdot \boldsymbol{\nabla}\zeta \ , \\[2em]
\boldsymbol{\nabla}\phi \longrightarrow -\dfrac{\partial}{\partial y}\left(\dfrac{\partial \phi}{\partial x}\right) + \dfrac{\partial}{\partial x}\left(\dfrac{\partial \phi}{\partial y}\right) = 0 \ ; \\[2em]
f\hat{\mathbf{z}} \times \mathbf{u} \longrightarrow -\dfrac{\partial}{\partial y}(-fv) + \dfrac{\partial}{\partial x}(fu) \\
\qquad = f\left(\dfrac{\partial u}{pdx} + \dfrac{\partial v}{\partial y}\right) + u\dfrac{\partial f}{\partial x} + v\dfrac{\partial f}{\partial y} \\
\qquad = \mathbf{u} \cdot \boldsymbol{\nabla} f \ ; \\[2em]
\mathbf{F} \longrightarrow -\dfrac{\partial F^x}{\partial y} + \dfrac{\partial F^y}{\partial x} \equiv \mathcal{F} \ .
\end{array} \right.$$

$$\frac{D\zeta}{Dt} = -\mathbf{u} \cdot \boldsymbol{\nabla} f + \mathcal{F}$$

Conservation of vorticity

Vorticity is conserved following a water parcel because of viscous or external forces or spatial variations in f

$$\oint \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{r} = \iint \zeta \mathrm{d}A$$

$$\frac{D\mathcal{C}}{Dt} = \oint \left[-\psi \nabla f + \boldsymbol{F}\right] \cdot \mathbf{d}\boldsymbol{r}$$

$$\frac{D\zeta}{Dt} = -\mathbf{u} \cdot \boldsymbol{\nabla} f + \mathcal{F}$$

Conservation of potential vorticity

Since $\partial_t f = 0$, the potential vorticity $q = \zeta + f$ is conserved

$$\frac{D\zeta}{Dt} + \boldsymbol{u} \cdot \nabla f = \mathcal{F} \qquad \Longrightarrow \qquad \frac{Dq}{Dt} = \mathcal{F}$$

When F=0 and there are no flux of potential vorticity at the boundary, $\boldsymbol{u} q \cdot \hat{\boldsymbol{n}} = 0$

$$\frac{d}{dt} \int\int dx\, dy\, q^n = 0$$

A univariate dynamical system

The dynamics depends only on streamfunction for prescribed forcing and rotation. The pressure does not appear in the barotropic vorticity equation. The continuity equation reduces the temporal evolution from second to first order.

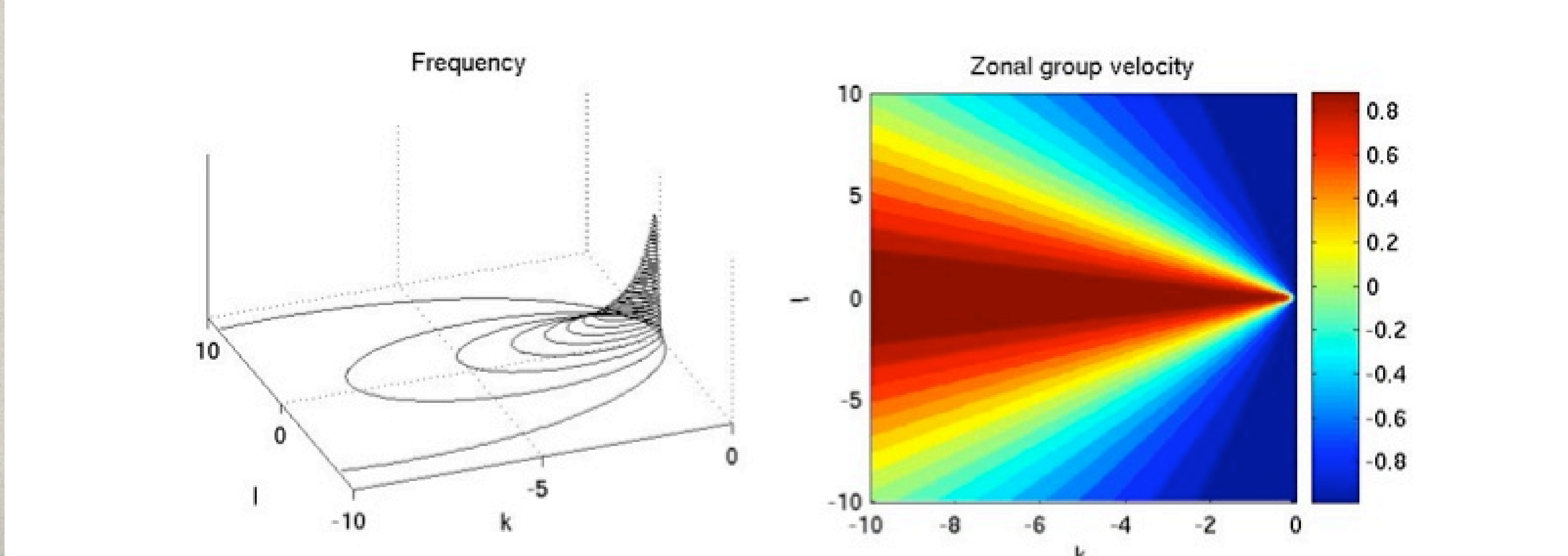
$$\nabla^2 \frac{\partial \psi}{\partial t} + J\left[\psi, f + \nabla^2 \psi\right] = \mathcal{F}$$

Rossby waves

$$\nabla^2 \frac{\partial \psi}{\partial t} + \beta_0 \frac{\partial \psi}{\partial x} \; = \; 0$$

$$\psi \; = \; \text{Real}\left(\psi_0 e^{i(kx+\ell y-\omega t)}\right) \quad \longrightarrow \quad \omega \; = \; -\frac{\beta_0 k}{k^2+\ell^2}$$

$$c_x = \frac{\omega}{k} = -\frac{\beta}{k^2+l^2} \qquad c_{gx} = \frac{\partial \omega}{\partial k} = \frac{\beta(k^2-l^2)}{k^2+l^2}$$



Rossby waves in the ocean

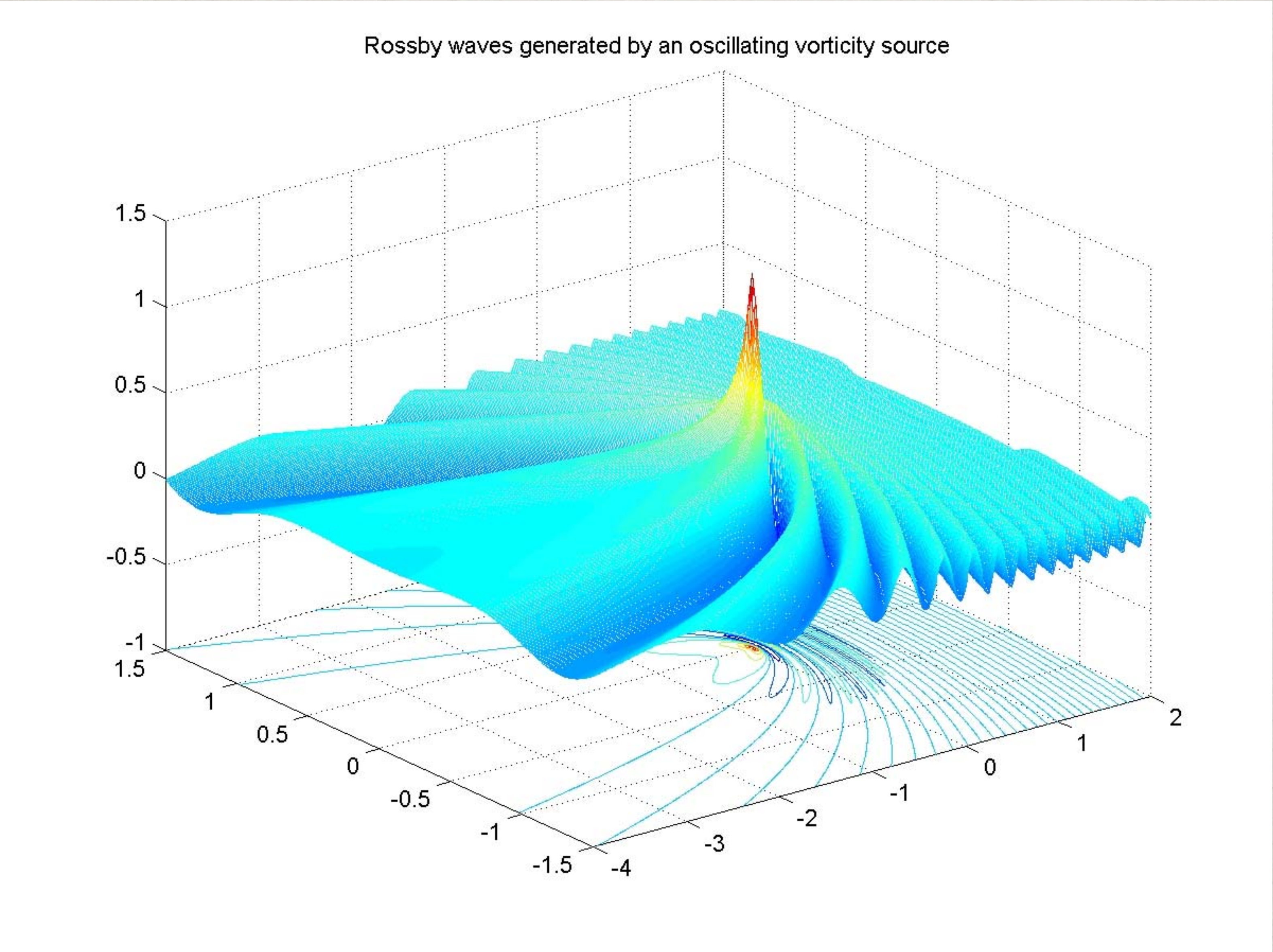


Fig. 14.11 Rossby wave pattern, Eqn. (14.21), created by an oscillating, small disturbance at the origin. Plotted is the pressure, or free-surface elevation η, or streamfunction ψ, at a particular time, as seen from the southwest (negative y, negative x). The parabolic wave-crests (see as contours on the base plane) sweep westward with time, closing in on the negative x-axis.

Rossby waves in the laboratory

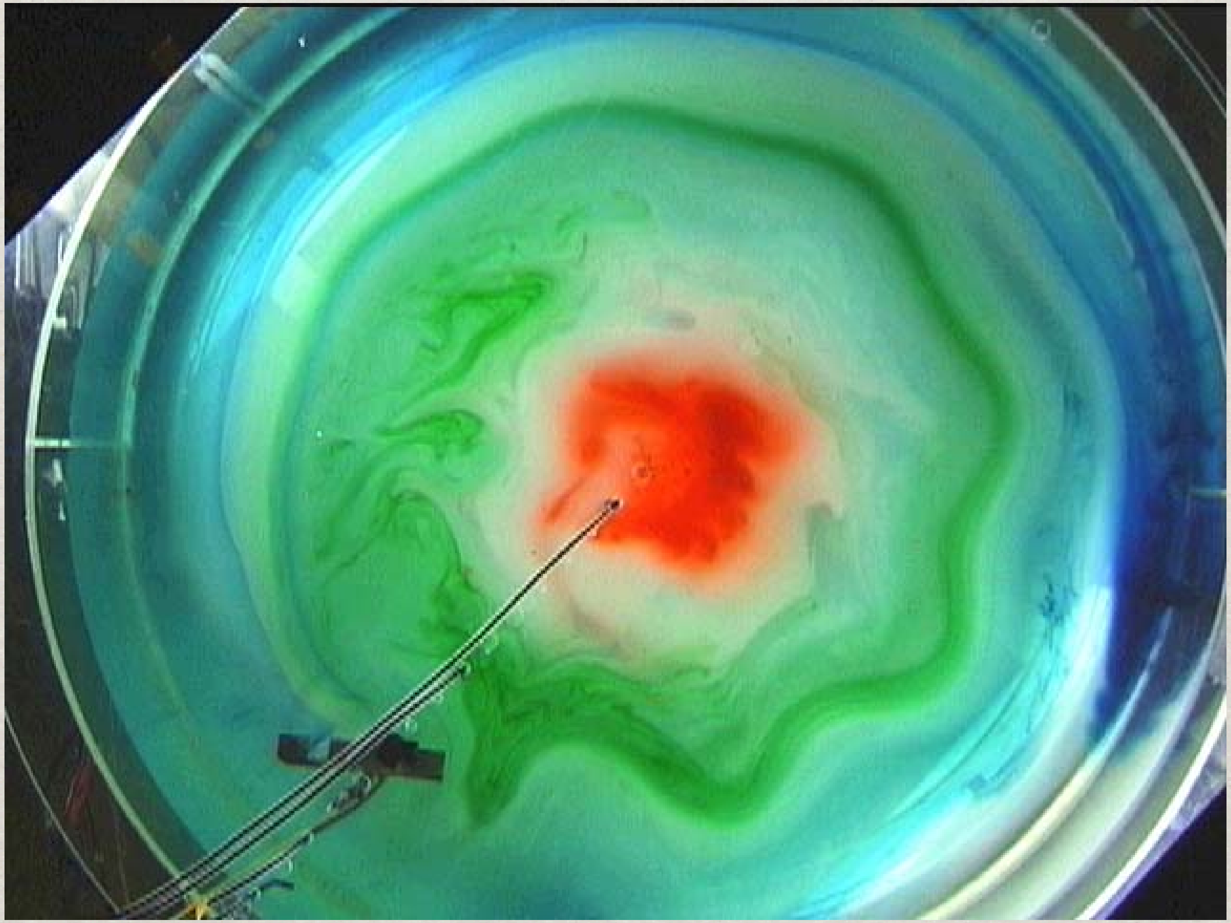


Figure 14.13. Rossby waves in the laboratory, as if viewed by a satellite above the North Pole. The wave source is at the lower left, and oscillating body. There is no pre-existing circulation, but the waves induce easterly flow at most latitudes, and westerly flow at the latitudes near the forcing (as seen in the dye

Rossby waves in the ocean

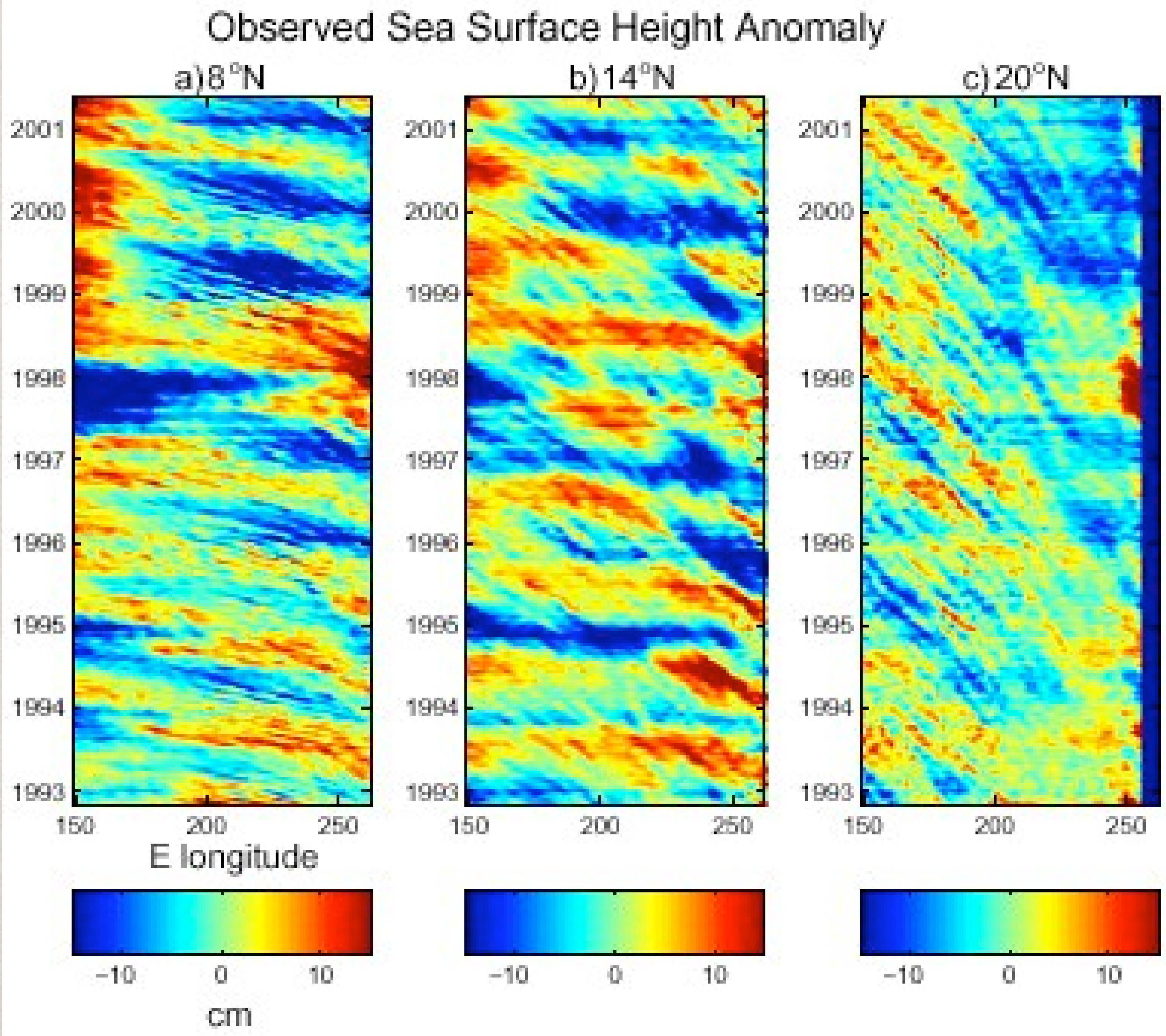


Figure 1. Time-longitude plots of SSH anomalies at (a) 8, (b) 14, and (c) 20°N from the TOPEX/POSEIDON altimeter. Units are meters.

Divergence equation

$$\nabla \cdot \left\{ \begin{array}{l} \dfrac{\partial \mathbf{u}}{\partial t} \longrightarrow \dfrac{\partial}{\partial x}\dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial y}\dfrac{\partial v}{\partial t} = 0 \\[2ex] (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} \longrightarrow \dfrac{\partial}{\partial x}\left(u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y}\right) + \dfrac{\partial}{\partial y}\left(u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y}\right) \\[2ex] \qquad = -2\left(\dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial y} - \dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial x}\right) \\[2ex] \qquad = \dfrac{\partial \mathbf{u}}{\partial t} = -2J\left[\dfrac{\partial \psi}{\partial x}, \dfrac{\partial \psi}{\partial y}\right] ; \\[2ex] -\boldsymbol{\nabla}\phi \longrightarrow -\nabla^2\phi ; \\[2ex] f\hat{\mathbf{z}} \times \mathbf{u} \longrightarrow \dfrac{\partial}{\partial x}(-fv) + \dfrac{\partial}{\partial y}(fu) = -\boldsymbol{\nabla}\cdot(f\boldsymbol{\nabla}\psi) ; \\[2ex] \mathbf{F} \longrightarrow \boldsymbol{\nabla}\cdot\mathbf{F} . \end{array} \right.$$

$$\nabla^2\phi = \boldsymbol{\nabla}\cdot(f\boldsymbol{\nabla}\psi) + 2J\left[\frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y}\right] + \boldsymbol{\nabla}\cdot\mathbf{F}$$

Geostrophic balance

After neglecting $\boldsymbol{\nabla} \cdot \mathbf{F}$ and using the following scaling estimates,

$$\mathbf{u} \sim V, \quad \mathbf{x} \sim L, \quad f \sim f_0, \quad \psi \sim VL, \quad \phi \sim f_0 VL, \quad \beta = \frac{df}{dy} \sim Ro \frac{f_0}{L}$$

for $Ro \ll 1$,

$$\nabla^2 \phi \;=\; f_0 \nabla^2 \psi [\, 1 + \mathcal{O}(Ro) \,] \quad \Rightarrow \quad \phi \;\approx\; f_0 \psi \; .$$

Gradient-wind balance

$$\nabla^2 \phi \;=\; \boldsymbol{\nabla} \cdot (f \boldsymbol{\nabla} \psi) + 2J \left[\frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y} \right]$$

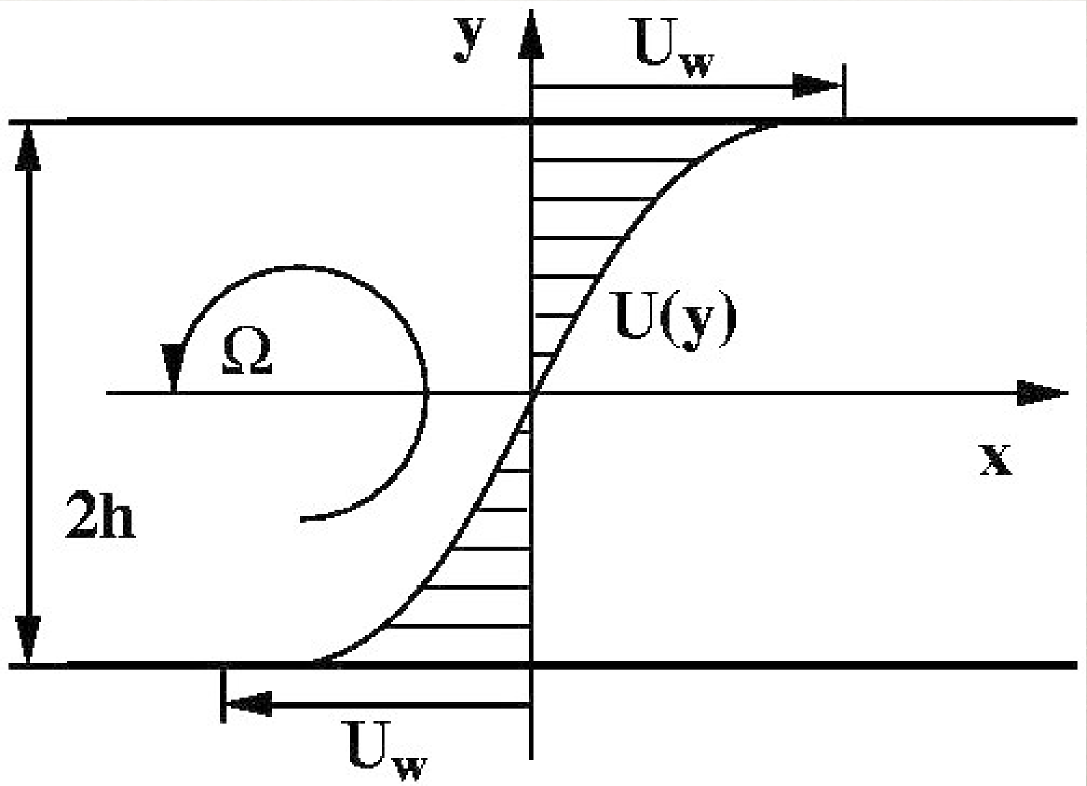
Exact for 2D motions, and highly accurate for 3D motions with $Ro \leq \mathcal{O}(1)$

Stationary solution: zonal shear flow

A parallel shear flow is an inviscid stationary solution

$$\mathbf{u}(\mathbf{x}, t) \;=\; U(y)\,\hat{\mathbf{x}}$$

$$J[\psi,\;q] \;=\; J\left[-\int^{y} U(y')\,dy',\;\; f(y) - \frac{dU}{dy}(y)\right] \;=\; 0$$

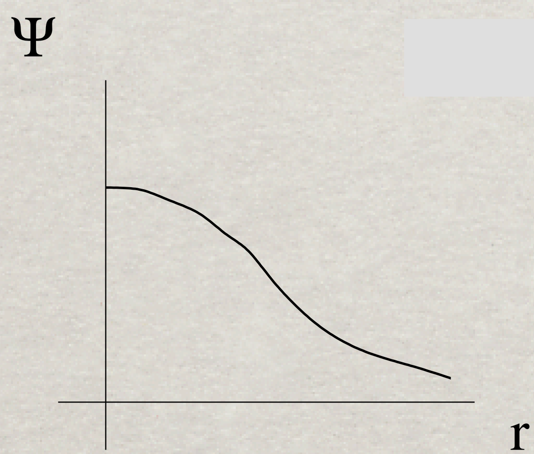


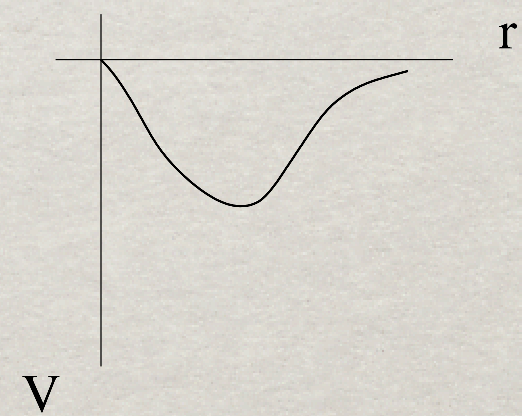
Stationary solution: axisymmetrix vortex flow

$$\mathbf{u} \;=\; \mathbf{U} \;=\; \hat{\mathbf{z}} \times \boldsymbol{\nabla}\psi \;\longrightarrow\; V \;=\; \frac{\partial \psi}{\partial r}, \quad U \;=\; 0$$

$$\zeta \;=\; \hat{\mathbf{z}} \cdot \boldsymbol{\nabla} \times \mathbf{u} \;\longrightarrow\; \zeta \;=\; \frac{1}{r}\frac{\partial}{\partial r}\left[rV\right] \;.$$

where U(r) is the radial velocity, V(r) is the azimuthal velocity.





Far-field generated by a 2D axisymmetric vortex

A monopole vortex whose vorticity, $\zeta(r)$, is restricted to a finite core region ($i.e.$, $\zeta = 0$ for all $r \geq r_*$) has a nearly universal structure to its velocity in the *far-field* region well away from its center. Integrating the last relation in

$$V(r) \;=\; \frac{1}{r}\int_0^r \zeta(r')r'\,dr'$$

For $r \geq r_*$, this implies that

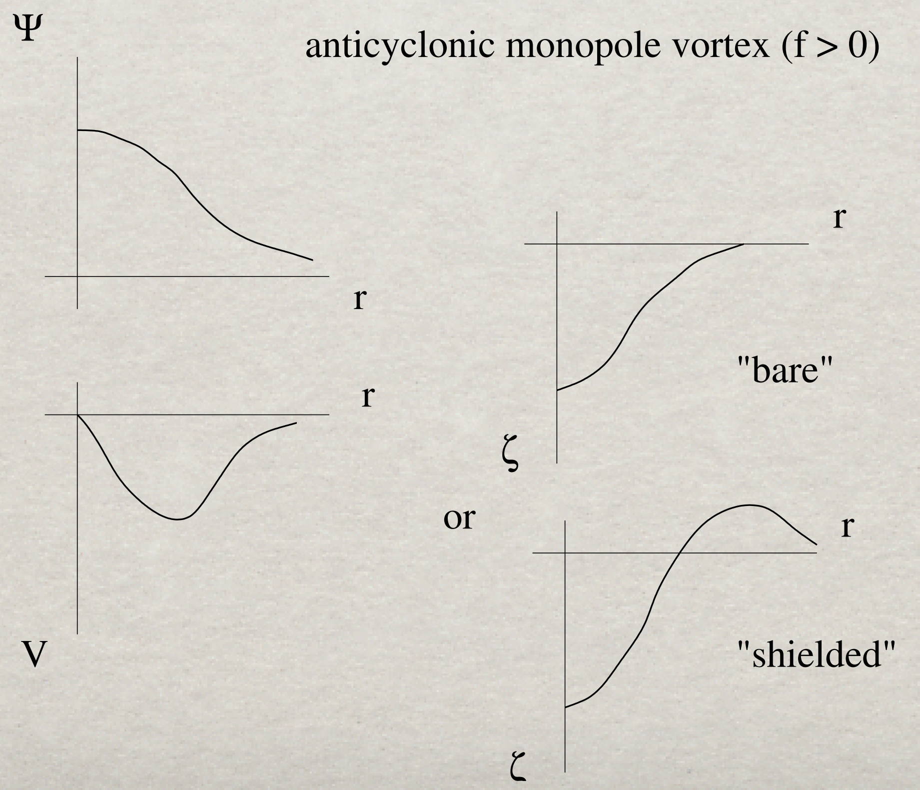
$$V(r) \;=\; \frac{C}{2\pi r}\;.$$

The associated far-field circulation, C, is

$$\begin{aligned} C(r) \;&=\; \int_{r \geq r_*} \mathbf{u} \cdot d\mathbf{r}' \\ &=\; \int_0^{2\pi} V(r) r\, d\theta \\ &=\; 2\pi r V(r) \\ &=\; 2\pi \int_0^{r_*} \zeta(r')r'\,dr'\;. \end{aligned}$$

Far-field generated by a 2D axisymmetric vortex

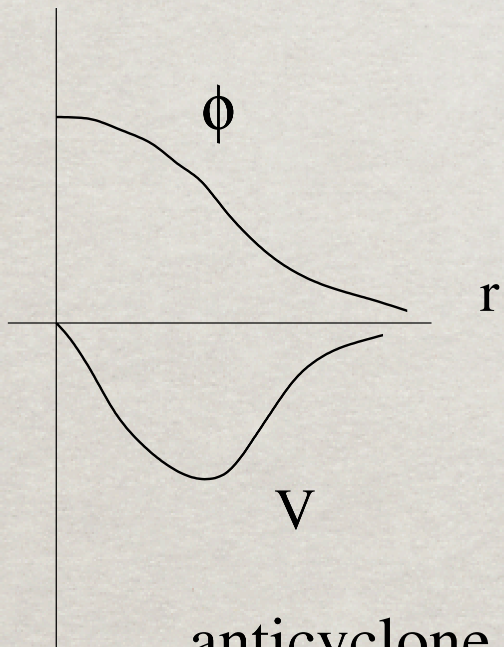
$$\begin{aligned}
\frac{\partial\psi}{\partial r} &= V \\
\Rightarrow \ \psi &= \int_0^r V\,dr' + \psi_0 \\
\Rightarrow \ \psi &\sim \psi_0 \quad \text{as} \quad r \to 0 \\
\Rightarrow \ \psi &\sim \frac{C}{2\pi}\ln r \quad \text{as} \quad r \to \infty\,.
\end{aligned}$$



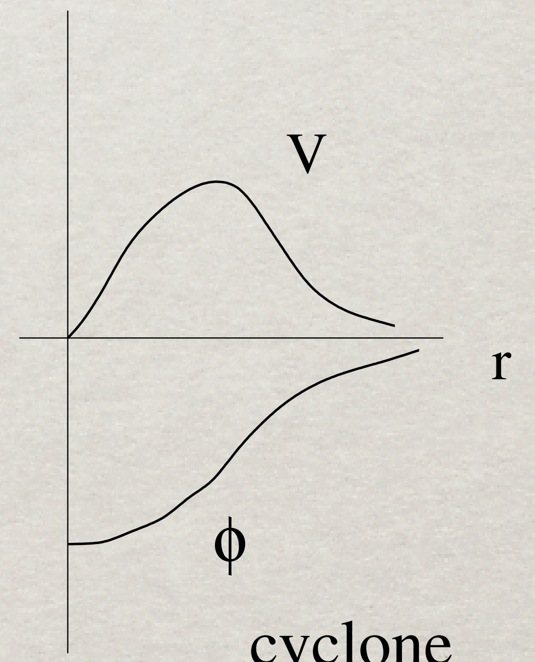
Cyclonic and anticyclonic vortices

<u>cyclonic</u>: $V > 0,\ \zeta > 0,\ C > 0,\ \psi < 0,\ \phi < 0.$

<u>anticyclonic</u>: $V < 0,\ \zeta < 0,\ C < 0,\ \psi > 0,\ \phi > 0.$



anticyclone



cyclone

Gradient-wind balance

The more general form of the divergence equation is the gradient-wind balance (3.42). For an axisymmetric state with $\partial_\theta = 0$,

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{\partial\phi}{\partial r}\right] = \frac{f_0}{r}\frac{d}{dr}\left[r\frac{\partial\psi}{\partial r}\right] + \frac{1}{r}\frac{d}{dr}\left[\left(\frac{\partial\psi}{\partial r}\right)^2\right] .$$

This can be integrated, $-(\int_r^\infty \cdot\, r\, dr)$, to obtain

$$\frac{\partial\phi}{\partial r} = fV + \frac{1}{r}V^2 . \qquad (@)$$

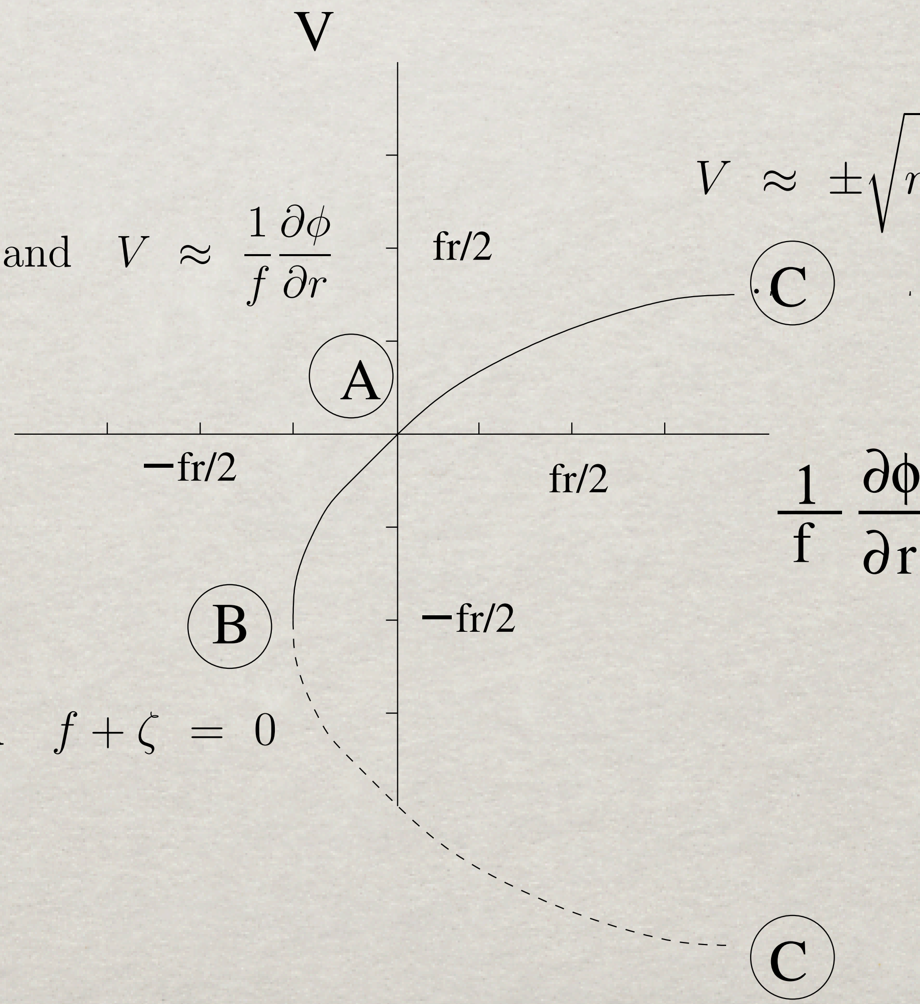
This expresses a radial force balance in a vortex among pressure-gradient, Coriolis, and centrifugal forces, respectively. (By induction it indicates that the third term in (3.42) is more generally the divergence of a centrifugal force along curved, but not necessarily circular, trajectories.) Equation (@) is a quadratic algebraic equation for V with solutions,

$$V(r) = -\frac{fr}{2}\left(1 \pm \sqrt{1 + \frac{4}{f^2 r}\frac{\partial\phi}{\partial r}}\right) .$$

$$\frac{\partial \phi}{\partial r} = fV + \frac{1}{r}V^2 \qquad\qquad V(r) = -\frac{fr}{2}\left(1 \pm \sqrt{1 + \frac{4}{f^2 r}\frac{\partial \phi}{\partial r}}\right)$$

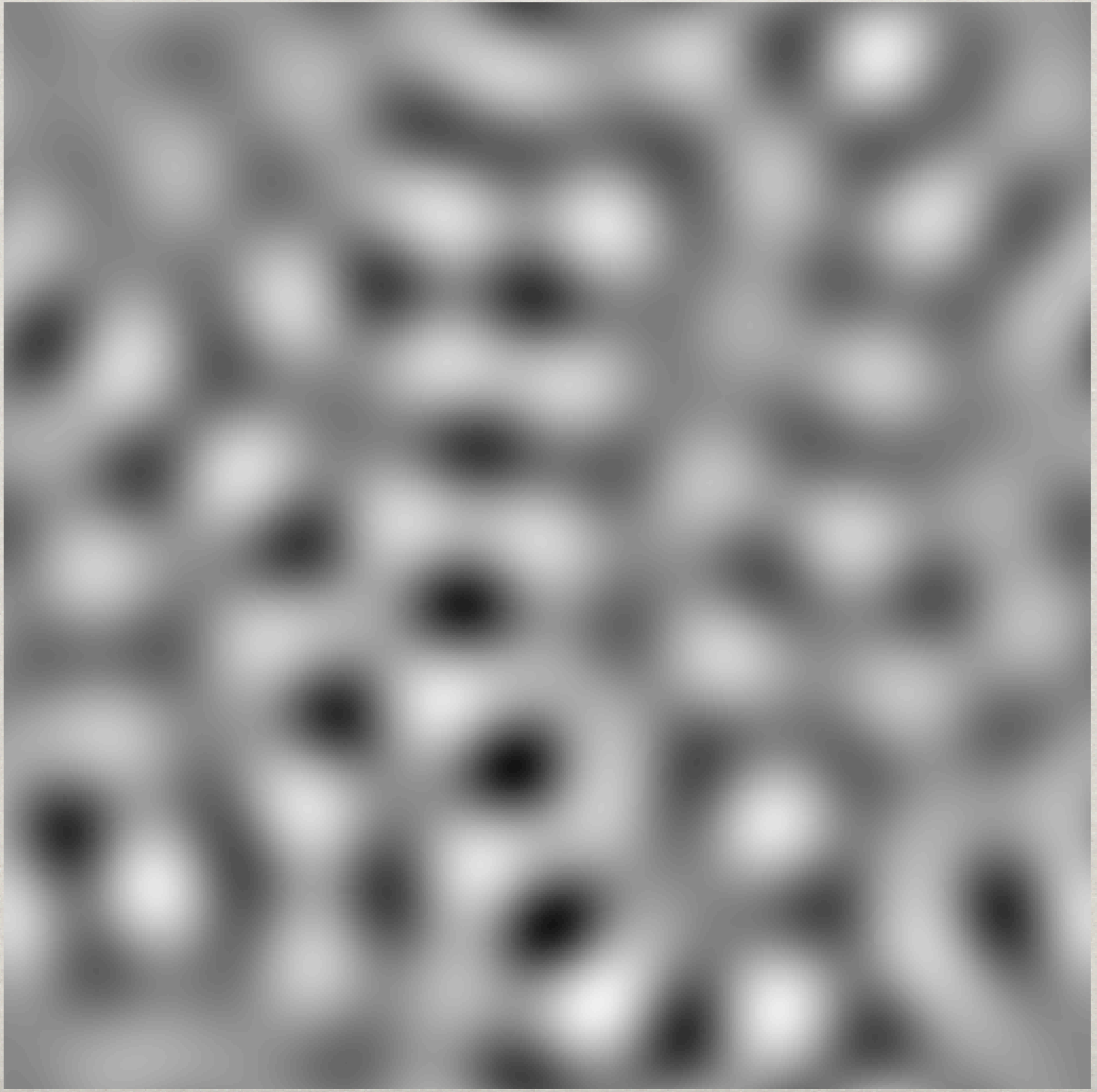
$$V \approx \pm\sqrt{r\frac{\partial \phi}{\partial r}} \quad (Ro \to \infty)$$

$$\frac{1}{f^2 r}\frac{\partial \phi}{\partial r} \to 0 \quad (Ro \to 0) \quad \text{and} \quad V \approx \frac{1}{f}\frac{\partial \phi}{\partial r}$$



$$V = -\frac{fr}{2}, \quad \frac{\partial \phi}{\partial r} = -\frac{f^2 r}{4}, \quad \text{and} \quad f + \zeta = 0$$

Vorticity in a 2D turbulent flow



Stationary solution: point vortex

Mathematical formulas for a point vortex located at $\mathbf{x} = \overline{\mathbf{x}}$ are

$$\begin{aligned}
\zeta &= C\delta(\mathbf{x} - \mathbf{x}_*) \\
V &= C/2\pi r \ , \\
\psi &= c_0 + C/2\pi \ \ln r \\
\mathbf{u} &= V\hat{\boldsymbol{\theta}} \ = \ V(-\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{y}}) \\
&= V(-\frac{y}{r}\hat{\mathbf{x}} + \frac{x}{r}\hat{\mathbf{y}}) \\
&= \frac{C}{2\pi r^2}(-y\hat{\mathbf{x}} + x\hat{\mathbf{y}})
\end{aligned}$$

the trajectory of a fluid parcel is generated from

$$\frac{d\mathbf{x}}{dt}(t) \ = \ \mathbf{u}(\mathbf{x}(t), t) \ , \qquad \mathbf{x}(0) \ = \ \mathbf{x}_0 \ .$$

Non stationary solutions: point vortices

$$\begin{aligned}
\zeta(\mathbf{x},t) &= \sum_{\alpha=1}^{N} C_\alpha \delta(\mathbf{x}-\mathbf{x}_\alpha) \\
\psi(\mathbf{x},t) &= \frac{1}{2\pi}\sum_{\alpha=1}^{N} C_\alpha \ln|\mathbf{x}-\mathbf{x}_\alpha| \\
\mathbf{u}(\mathbf{x},t) &= \frac{1}{2\pi}\sum_{\alpha=1}^{N} \frac{C_\alpha}{|\mathbf{x}-\mathbf{x}_\alpha|^2}[-(y-y_\alpha)\hat{\mathbf{x}}+(x-x_\alpha)\hat{\mathbf{y}})]
\end{aligned}$$

$$\begin{aligned}
\dot{x}_\alpha &= -\frac{1}{2\pi}\sum_{\beta}{}' \frac{C_\beta}{|\mathbf{x}_\alpha-\mathbf{x}_\beta|^2}(y_\alpha-y_\beta) \\
\dot{y}_\alpha &= +\frac{1}{2\pi}\sum_{\beta}{}' \frac{C_\beta}{|\mathbf{x}_\alpha-\mathbf{x}_\beta|^2}(x_\alpha-x_\beta)
\end{aligned}$$

Point vortex trajectories

Show matlab simulations