# 12.803
# Quasi-Balanced Motions in the Ocean and atmosphere

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# Class schedule

- Class schedule: M-W 10.30-12.00
- Bi-weakly homework assignments (60%)
- Final exam (30%)
- Questions during class (10%)
- Web page (http://mit.edu/~raffaele/www/Classes/12.803/index.html)
- Coordination with 12.804 (lab class)

# Syllabus

1. Fundamental conservation principles for large-scale flow
    - Mass and momentum conservations
    - Beta plane
    - Hydrostatic and geostrophic balance

2. Barotropic vorticity equation
    - Conservation of potential vorticity
    - The invertibility principle

3. Shallow water equations
    - Conservation of potential vorticity
    - Balance and inversion
    - Geostrophic adjustment
    - Separation of flow into balanced and unbalanced parts
    - Quasi-geostrophic and higher-order balance equations

4. Simplified equations for the atmospheres and oceans
    - Thermodynamics
    - Traditional approximation
    - Quasi-geostrophic equations
    - The omega equation
    - The Sawyer-Eliassen equation

1. Stability theory
    - Linear stability theory
    - The Rayleigh and Fjrtoft theorems
    - Non-normal mode instability

2. Barotropic Instability
    - Kelvin Helmholtz instability
    - Parallel shear flow

3. Baroclinic Instability
    - The Eady model
    - The Charney model
    - The Phillips models
    - The Charney-Stern theorem
    - Ageostrophic baroclinic instability

4. Wave-mean flow interactions
    - The Eliassen-Palm theorem
    - The Transformed Eulerian Mean

1. Turbulence
    - Three dimensional turbulence
    - Two dimensional turbulence
    - Coherent structures

2. Geostrophic turbulence
    - Effect of differential rotation in two dimensional turbulence
    - Stratified geostrophic turbulence
    - Macroturbulence in the atmosphere
    - Macroturbulence in the ocean

# TEXTBOOKS

- R. Salmon, Lectures on Geophysical Fluid Dynamics (Oxford University Press)

- A. E. Gill, Atmosphere-Ocean Dynamics (Academic Press)

- J. Pedlosky, Geophysical Fluid Dynamics (Springer-Verlag)

- G. Vallis, Atmospheric and Oceanic Fluid Dynamics, available from www.princeton.edu/~gkv/aofd (to be published by Cambridge University Press)

# Lecture I

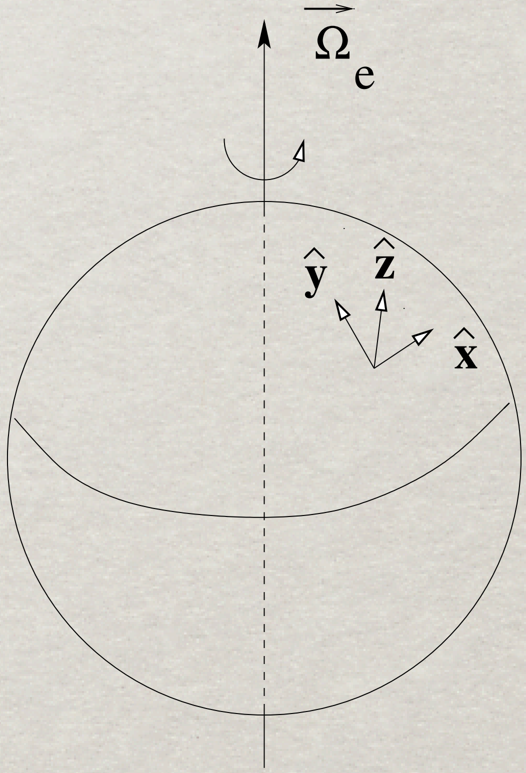
- Equation of motions for a rotating fluid

$$\frac{Du}{Dt} - 2\Omega_v v + 2\Omega_h w = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{Dv}{Dt} + 2\Omega_v u = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{Dw}{Dt} - 2\Omega_h u = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\boldsymbol{u} = 0$$




N.B. Valid under the shallow fluid approximation r=R+z

# Traditional approximation

$$\boldsymbol{\Omega} = (0, \Omega\cos\theta, \Omega\sin\theta)$$

$$\begin{aligned}
\tfrac{Du}{Dt} - 2\Omega_v v + \xcancel{2\Omega_h w} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} \\
\tfrac{Dv}{Dt} + 2\Omega_v u &= -\frac{1}{\rho}\frac{\partial p}{\partial y} \\
\tfrac{Dw}{Dt} - \xcancel{2\Omega_h u} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \\
\tfrac{D\rho}{Dt} + \rho\nabla\cdot\boldsymbol{u} &= 0
\end{aligned}$$

# Beta-plane approximation

$$\begin{aligned} f = 2\Omega_v = 2\Omega\sin\theta &\approx 2\Omega\sin\theta_0 + 2\Omega\cos\theta_0(\theta-\theta_0) \\ &\approx 2\Omega\sin\theta_0 + \frac{2\Omega\cos\theta_0}{R}(y-y_0) \\ &\approx f_0 + \beta(y-y_0) \end{aligned}$$

$$\begin{aligned} \tfrac{Du}{Dt} - (f_0+\beta y)v &= -\frac{1}{\rho}\frac{\partial p}{\partial x} \\ \tfrac{Dv}{Dt} + (f_0+\beta y)u &= -\frac{1}{\rho}\frac{\partial p}{\partial y} \\ \tfrac{Dw}{Dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \\ \tfrac{D\rho}{Dt} + \rho\nabla\cdot\boldsymbol{u} &= 0 \end{aligned}$$

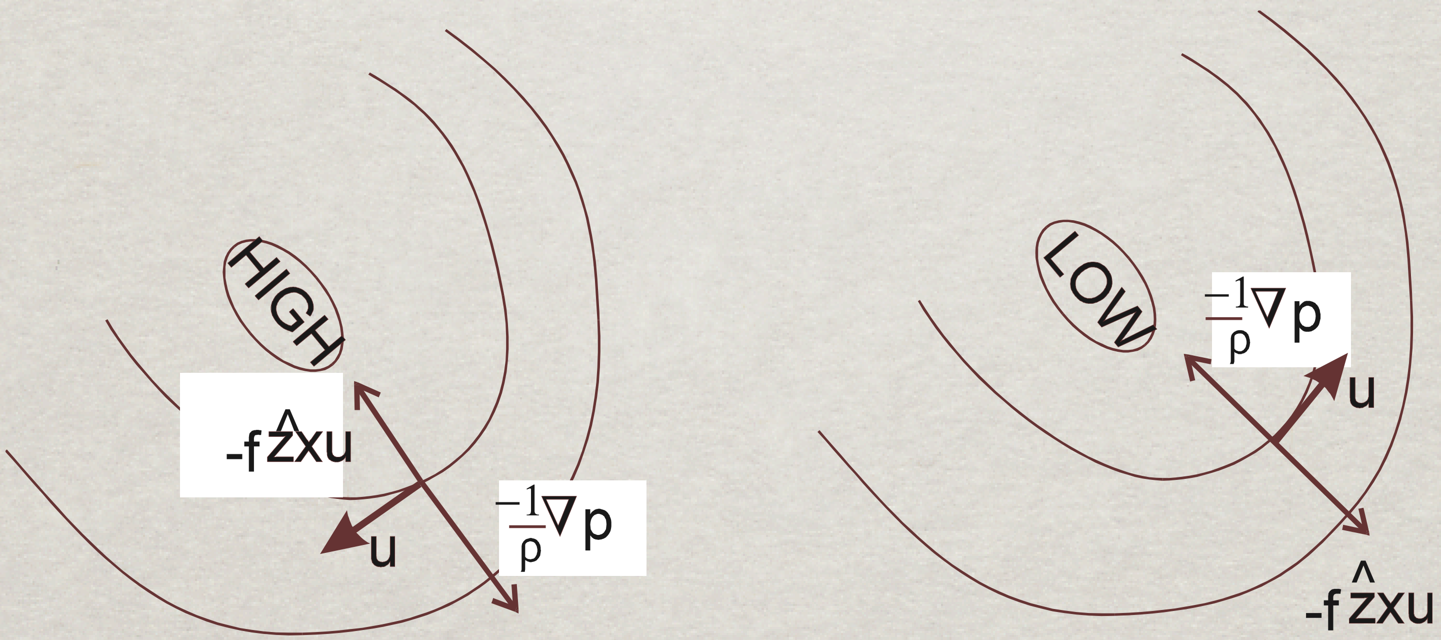
# Hydrostatic and geostrophic balance

$$
\begin{aligned}
-\rho f v &= -\frac{\partial p}{\partial x} \\
\rho f u &= -\frac{\partial p}{\partial y} \\
0 &= -\frac{\partial p}{\partial z} - \rho g
\end{aligned}
$$

Neglecting variations of density and rotation

$$
\psi = \frac{p}{\rho_0 f_0}, \qquad w_z = 0
$$

# Geostrophic flow and pressure field



# Geostrophic balance in pressure coordinates

## Change of coordinates

$$\begin{aligned}\frac{\partial p}{\partial x} &= \frac{\partial(p,y,z)}{\partial(x,y,z)} \\ &= \frac{\partial(p,y,z)}{\partial(x,y,p)}\frac{\partial(x,y,p)}{\partial(x,y,z)} \\ &= \left(-\frac{\partial z}{\partial x}\right)(-\rho g)\end{aligned}$$

$$\begin{aligned}\left.\frac{\partial p}{\partial x}\right)_z &= \left.\frac{\partial p}{\partial x}\right)_p - \left.\frac{\partial z}{\partial x}\right)_p \frac{\partial p}{\partial z} \\ &= \left(-\frac{\partial z}{\partial x}\right)(-\rho g)\end{aligned}$$

## Geostrophic balance

$$\begin{aligned}-fv &= -g\frac{\partial z}{\partial x} \\ fu &= -g\frac{\partial z}{\partial y} \\ \frac{\partial z}{\partial p} &= -\frac{1}{\rho g}\end{aligned}$$

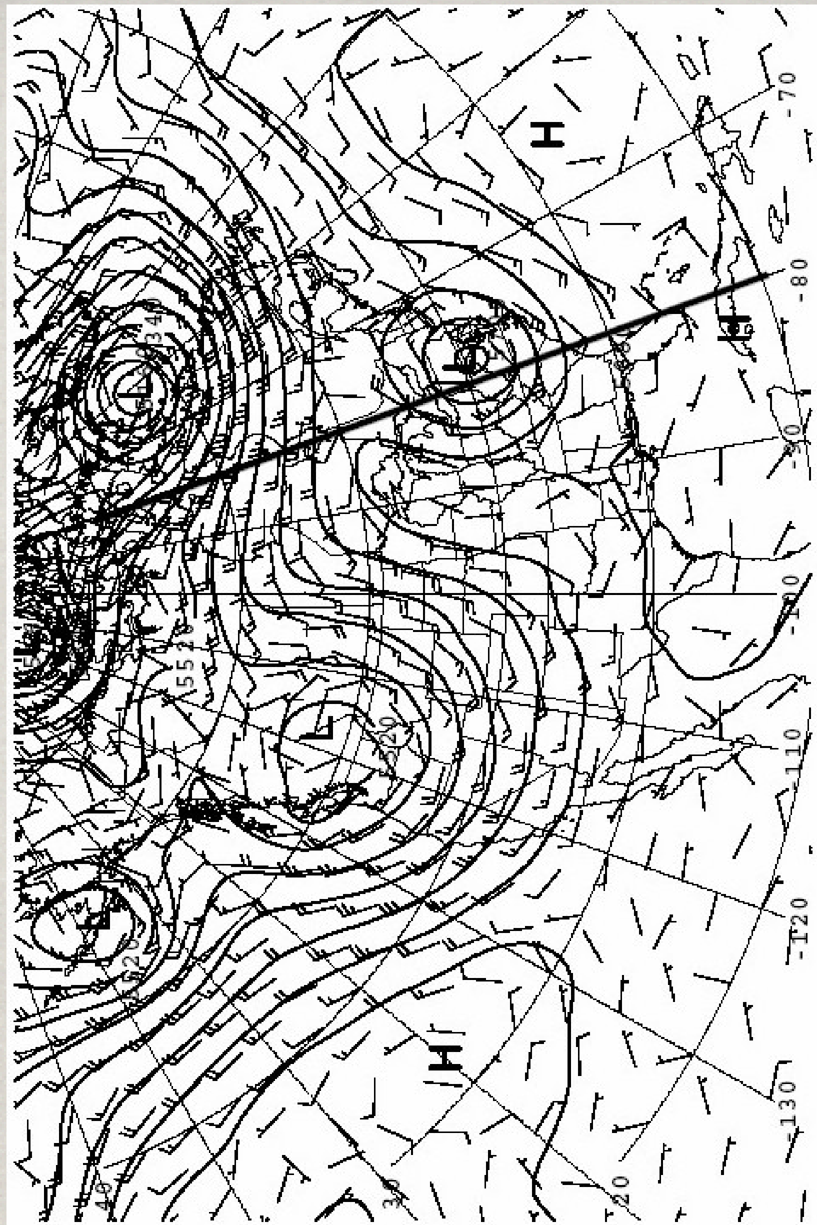



Figure 7.4: The 500 mbar wind and geopotential height field at 12GMT on June 21st, 2003. The wind blows away from the quiver: one full quiver denotes a speed of $10\,\mathrm{m\,s^{-1}}$, one half-quiver a speed of $5\,\mathrm{m\,s^{-1}}$. The geopotential height is contoured every 60 m. Centers of high and low pressure are marked $H$ and $L$. The position marked $A$ is used as a check on geostrophic balance. The thick black lines marks the position of the meridional section shown in Fig.7.21 at 80°W extending from 20°N to 70°N. This section is also marked on Fig.7.20.

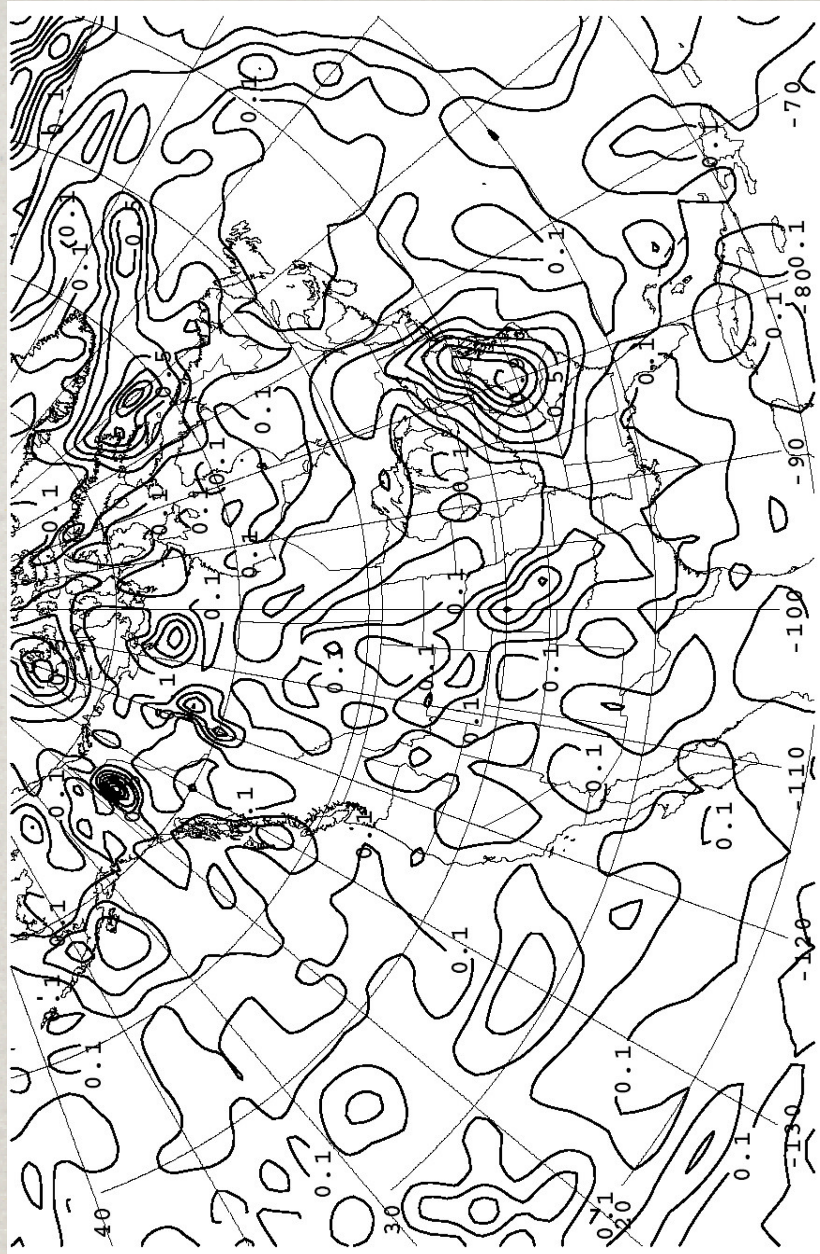


Figure 7.5: The Rossby number for the 500 mbar flow at 12GMT on June 21st, 2003, the same time as Fig.7.4. Note that $R_o \sim 0.1$ over most of the region but can approach 1 in strong cyclones, such as the low centered over 80°W, 40°N.

# Thermal wind balance in the atmosphere

## Equation of state for an ideal gas

$$\frac{p}{\rho} = RT$$

## Thermal wind balance

$$\begin{aligned} -f\frac{\partial v}{\partial p} &= \frac{\partial}{\partial x}\left(\frac{1}{\rho}\right) = \frac{R}{p}\frac{\partial T}{\partial x} \\ f\frac{\partial u}{\partial p} &= \frac{\partial}{\partial y}\left(\frac{1}{\rho}\right) = -\frac{R}{p}\frac{\partial T}{\partial y} \end{aligned}$$

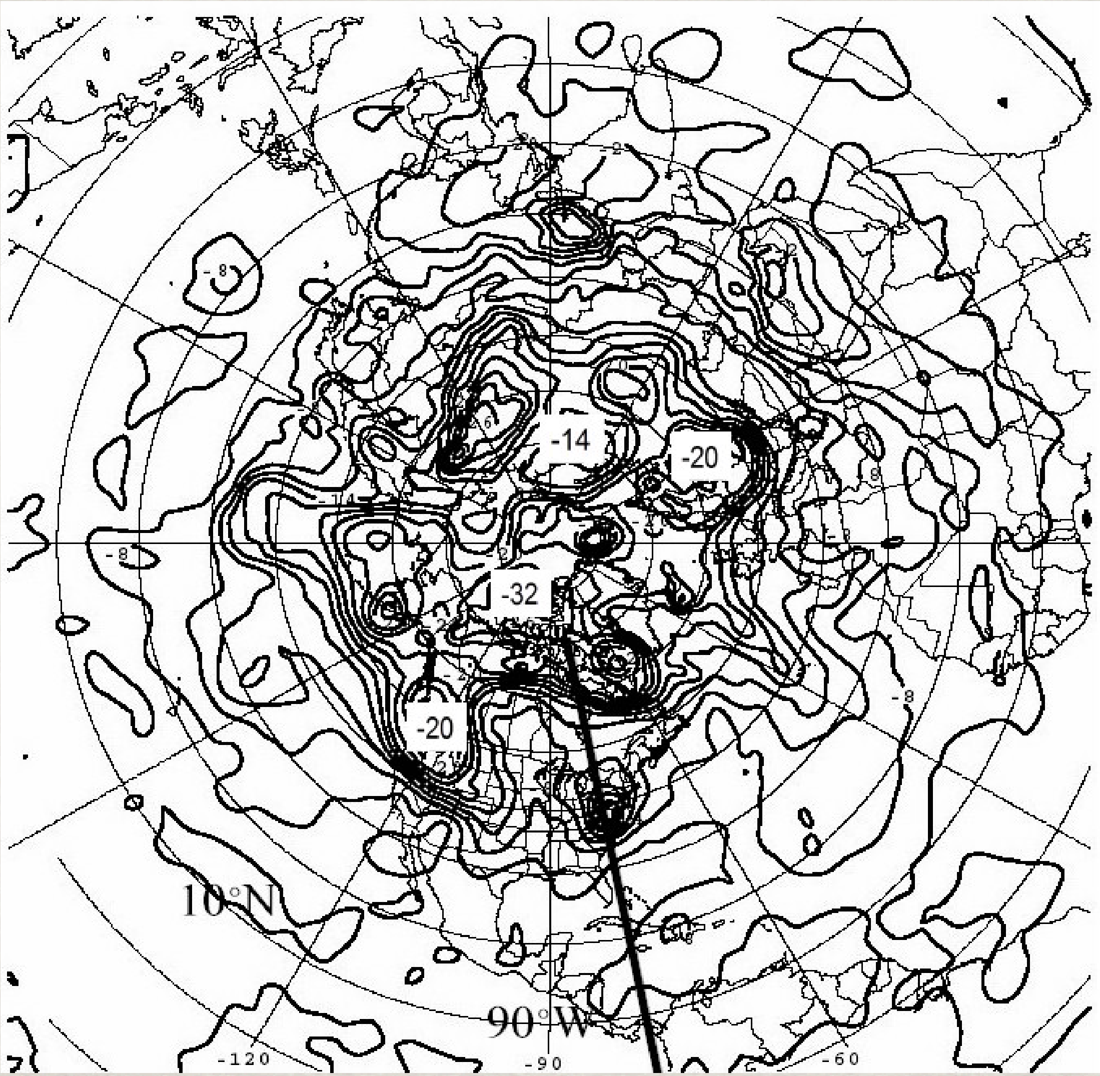



Figure 7.20: The temperature, $T$, on the 500 mbar surface at 12GMT on June 21st, 2003 — c.i = 2 °C — the same time as Fig.7.4. The thick black lines marks the position of the meridional section shown in Fig.7.21 at 80°W extending from 20°N to 70°N. The coldest temperatures over the pole get as low as −32 °C.

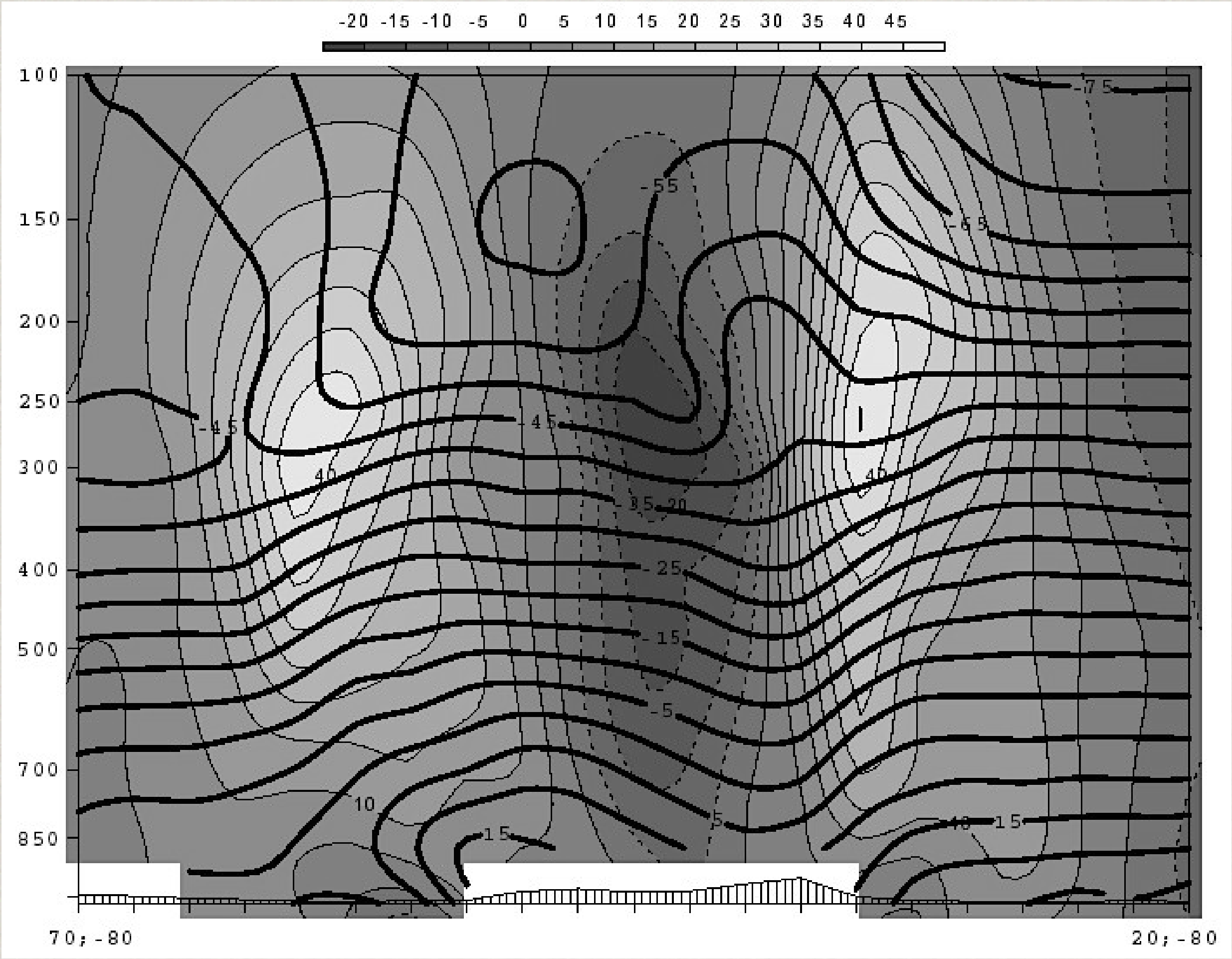



Figure 7.21: A cross section of zonal wind, $u$ (grey-scale and thin contours every $5\,\mathrm{m\,s^{-1}}$, and potential temperature, $T$ (thick contours every $5\,^{\circ}\mathrm{C}$) through the atmosphere at 80°W extending from 20°N to 70°N on June 21st, 2003 on at 12GMT, as marked on Figs.7.20 and 7.4. Note that $\frac{\partial u}{\partial z} > 0$ in regions where $\frac{\partial T}{\partial y} < 0$ and visa-versa.

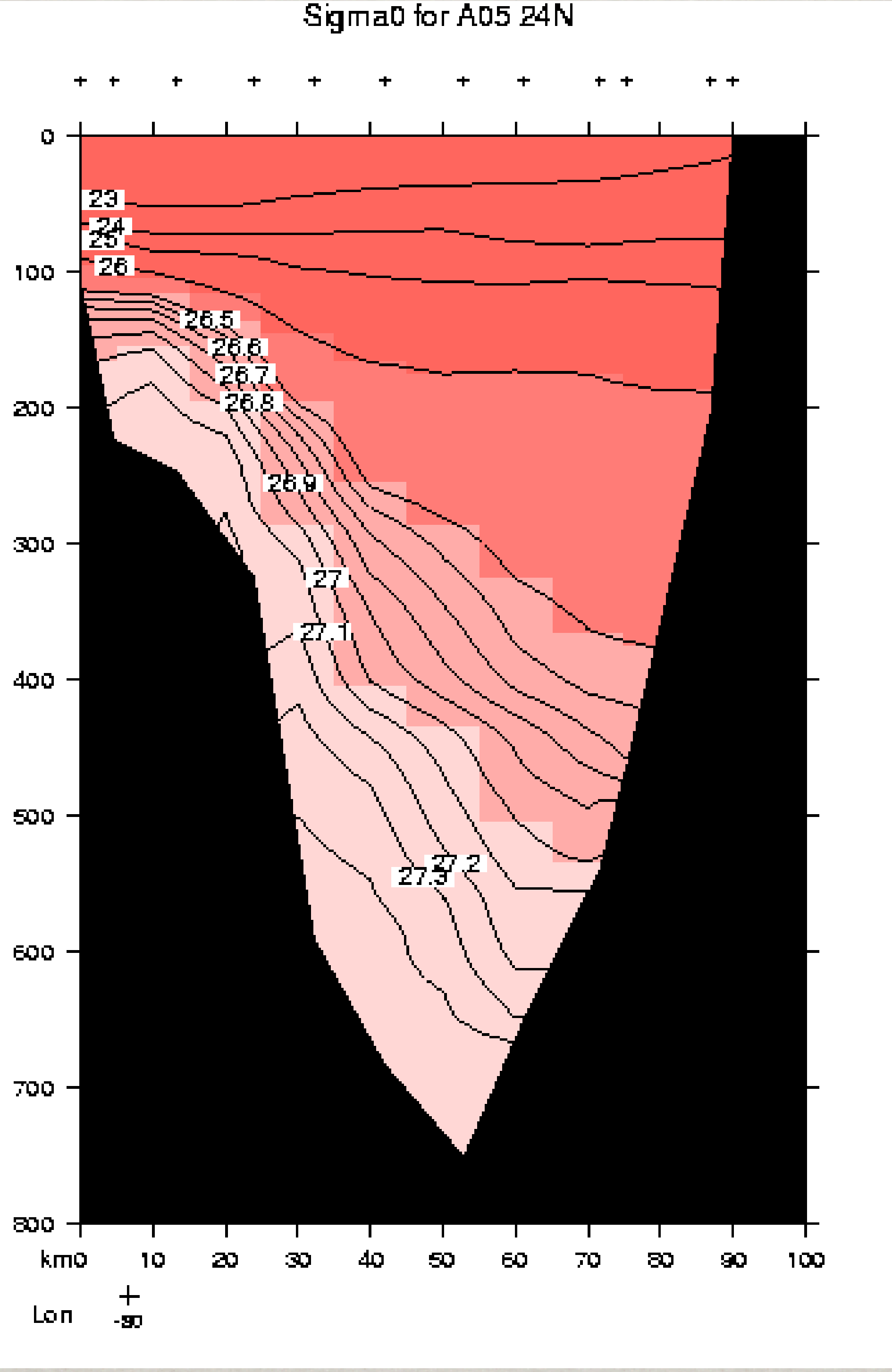
# Thermal wind balance in the ocean

## Thermal wind balance

$$
\begin{aligned}
f\frac{\partial(\rho v)}{\partial z} &= -g\frac{\partial \rho}{\partial x} \\
f\frac{\partial(\rho u)}{\partial z} &= +g\frac{\partial \rho}{\partial y} \\
\frac{\partial(\rho f u)}{\partial x} + \frac{\partial(\rho f v)}{\partial y} &= 0
\end{aligned}
$$

## In the ocean density fluctuations are less than 5%

$$
\begin{aligned}
\frac{\partial v}{\partial z} &= -\frac{g}{f\rho_0}\frac{\partial \sigma}{\partial x} \\
\frac{\partial u}{\partial z} &= +\frac{g}{f\rho_0}\frac{\partial \sigma}{\partial y} \\
\beta v &= f\frac{\partial w}{\partial z}
\end{aligned}
$$


Sigma0 for A05 24N
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