Homework assignment 5

Problem 1: Barotropic jet

Obtain the stability properties of the triangular jet, with a basic state velocity given by,

$$U(y) = \begin{cases} 0 & y \geq 1 \\ 1 - |y| & -1 \leq y \leq 1 \\ 0 & y \leq -1 \end{cases} \tag{1}$$

Inparticular, obtain the eigenfunctions and eigenvalues of the problem, and show that each eigenfunction is either even or odd. Perturbations with even ψ' are known as *sinuous modes* and those with odd ψ' are *varicose modes*. Show that sinuous waves are unstable for sufficiently long wavelengths in the y-direction, but that all varicose modes are stable.

Problem 2: Generalized Eady model

In class, we discussed two classical baroclinic instability problems applied to the atmosphere, the Charney and Eady problems. Both are based on quasi–geostrophic theory on a β–plane where linearized perturbations on a basic zonal flow, $u_0(z)$, are described by the equation for conservation of pseudo–potential vorticity:

$$(\frac{\partial}{\partial t} + u_0 \frac{\partial}{\partial x})q' + \psi'_x \overline{q}_y = 0$$

where

$$\overline{q}_y = \beta - \frac{f_0^2}{\overline{\rho}} \frac{\partial}{\partial z} \left(\frac{\overline{\rho}}{N^2} \frac{\partial}{\partial z} u_0 \right)$$

and

$$q' = \nabla^2 \psi' + \frac{f_0^2}{\overline{\rho}} \frac{\partial}{\partial z} \left(\frac{\overline{\rho}}{N^2} \frac{\partial}{\partial z} \psi' \right)$$

For the Charney model, the density is assumed to decrease exponentially with height (i.e., a constant scale height H), the vertical shear of the background flow, du_0/dz and N^2 are assumed constant, and there is a rigid lower boundary. The Eady model takes the even more drastic assumption of setting β to be zero. In addition, the full Boussinesq approximation is made, which is equivalent to letting H be much larger than the depth of the system. In addition to a rigid lower boundary at $z = 0$, a rigid upper boundary is imposed at some $z = h$ such that the necessary conditions for the Charney–Stern (generalized Rayleigh) theorem can be met.

Although these approximations may seem extreme, observations over the past several years suggest that the atmosphere has much in common with the Eady problem in that $\overline{q}_y$ is small (and possibly zero) in the midlatitude troposphere. Of course, β is not zero in the real atmosphere, so if we ignore meridional variations in the zonal flow, the vanishing PV gradients must come from variations in static stability and or in shear with height must balance β.

For the following problems, assume a Boussinesq fluid and follow the original Eady model in replacing the tropopause with a rigid upper boundary.

a) Consider the case where the shear is constant but the static stability N^2 varies with height. Find how it must vary to cancel exactly the contribution of β to the meridional gradient of $\overline{q}$.

b) Repeat (a) for the case where N^2 is constant but the shear varies with height.

c) Estimate, using typical tropospheric conditions, the quantitative change is u_0 or N^2 needed to cancel β (as compared to a constant shear and N^2 profile). Here consider the top at $\simeq 10\ km$, a mean shear of $3\ m/s/km$ and a mean N^2 of 10^{-4}.

d) In the case of constant N^2, assuming a normal mode form $\psi' = \Psi(z)\exp(\imath kx + \imath ly - \imath\omega t)$, solve the eigenvalue problem for c (using appropriate boundary conditions). Find the dispersion relation and the vertical structure Ψ. How do the results differ from the original Eady problem?

Problem 3: Effect of topography in the Eady problem

In the classical Eady problem the upper and lower boundaries are flat. Repeat the analysis of the Eady instability in a domain where the topography at the bottom has a linear slope, i.e. the two vertical boundaries are at $z = \gamma y$ and $z = D$. Assume that the slope is of order Rossby. In order to simplify the algebra use periodic boundary conditions in x and y and use modal solutions of the form $\psi' = \Psi(z)\exp(\imath kx + \imath ly - \imath kct)$. Does it make a difference whether the ground is inclided in the same or opposite sense than the isopycnals in the basic state?