Homework assignment 3

Problem 1

Show that the general definition of the Brunt–Väisälä frequency

$$N^2 = -g\frac{1}{\rho}\frac{\partial}{\partial z}\rho - \frac{g^2}{c_s^2}$$

where c_s^2 is the speed of sound, can be written as

$$N^2 = g\frac{\partial}{\partial z}\ln\theta$$

for an ideal gas.

Use this to find N^2 for a hydrostatic atmosphere with a temperature structure $T(z)$. What is it for an isothermal atmosphere?

Problem 2

The Boussinesq equations used to describe ocean motions are,

$$\frac{D}{Dt}\mathbf{u} + f\hat{\mathbf{k}} \times \mathbf{u} = -\nabla p + b\hat{\mathbf{k}}$$

$$\nabla \cdot \mathbf{u} = 0$$

with the buoyancy evolving according to

$$\frac{D}{Dt}b + N^2 w = 0.$$

You can find a derivation of these equations in section 2.16 of Salmon's book. Note that I use b to denote density fluctuations, while Salmon uses θ.

Derive the quasi-geostrophic approximation for this system of equations. Follow the same steps described in class for the primitive equations in pressure coordinates. (1) Assume that the motions are close to geostrophic balance. (2) Derive the momentum and mass conservation equations at order Rossby, assuming that variations of f with latitude are small. (3) Derive the vorticity equation from these order Rossby equations. (4) Derive the buoyancy equation at order Rossby. (5) Use the order Rossby buoyancy equation to eliminate w_z from the vorticity equation. (6) Write the conservation equation for the quasi-geostrophic potential vorticity in the ocean.

Problem 3

For an adiabatic gas, it is useful at times to consider θ as the vertical coordinate (i.e., x, y, z, $t \to x'$, y', $\xi \equiv \theta$, t'). This problem shows the close relationship between the θ–coordinate equations and the shallow water equations.

1) Argue that, in θ–coordinates, the "vertical" velocity $\omega = 0$ in the absence of heating.

2) Show that the horizontal momentum equations become

$$\frac{D}{Dt}\mathbf{u} + f\hat{\mathbf{k}} \times \mathbf{u} = -\nabla\Pi$$

with

$$\Pi = \phi + c_p T$$

Hint — how is dp related to dT when $d\theta = 0$? $c_p \ln\theta = c_p \ln T - R\ln p$

3) The conservation of mass equation can be written

$$\frac{D}{Dt}h + h\nabla \cdot \mathbf{u} = 0$$

with $h = \phi_\theta/\alpha$. Show that the potential vorticity

$$q = \frac{v_x - u_y + f}{h}$$

is conserved. Note that the advection operator is two–dimensional.

4) Finally, we need to link h and Π. From the change of coordinates and the hydrostatic relationship, convince yourself that

$$h = \frac{\phi_\theta}{\alpha} = -\frac{\partial}{\partial\theta}p$$

Use the definition of potential temperature to write p as a function of T/θ. Then show that the hydrostatic equation reduces to

$$\Pi_\theta = c_p\frac{T}{\theta}$$

This leads to an expression for h in terms of Π.