Homework assignment 1

Problem 1

The equation for mass conservation in (x, y, z, t) coordinates is,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0. \tag{1}$$

Using the approach described in class, derive the mass conservation equation in (x, y, θ, t) coordinates,

$$\frac{\partial}{\partial t}\left(\rho \frac{\partial z}{\partial \theta}\right) = ? \tag{2}$$

where θ is any scalar that is conserved on fluid particles, $D\theta/Dt = 0$. You might want to read the first chapter of Rick Salmon's book before answering on this question.

Problem 2

Consider a pendulum mounted on a spring subject to a uniform gravitational acceleration g, as shown in figure 1. Except for a small amount of damping applied to the radial motion, the system is idealized as frictionless. The equations of motion, and the equations for the displacement of the pendulum in the radial and azimuthal directions, are,

$$\frac{dv}{dt} = -\frac{uv}{r} - g\sin\theta, \tag{3}$$

$$\frac{d\theta}{dt} = \frac{v}{r}, \tag{4}$$

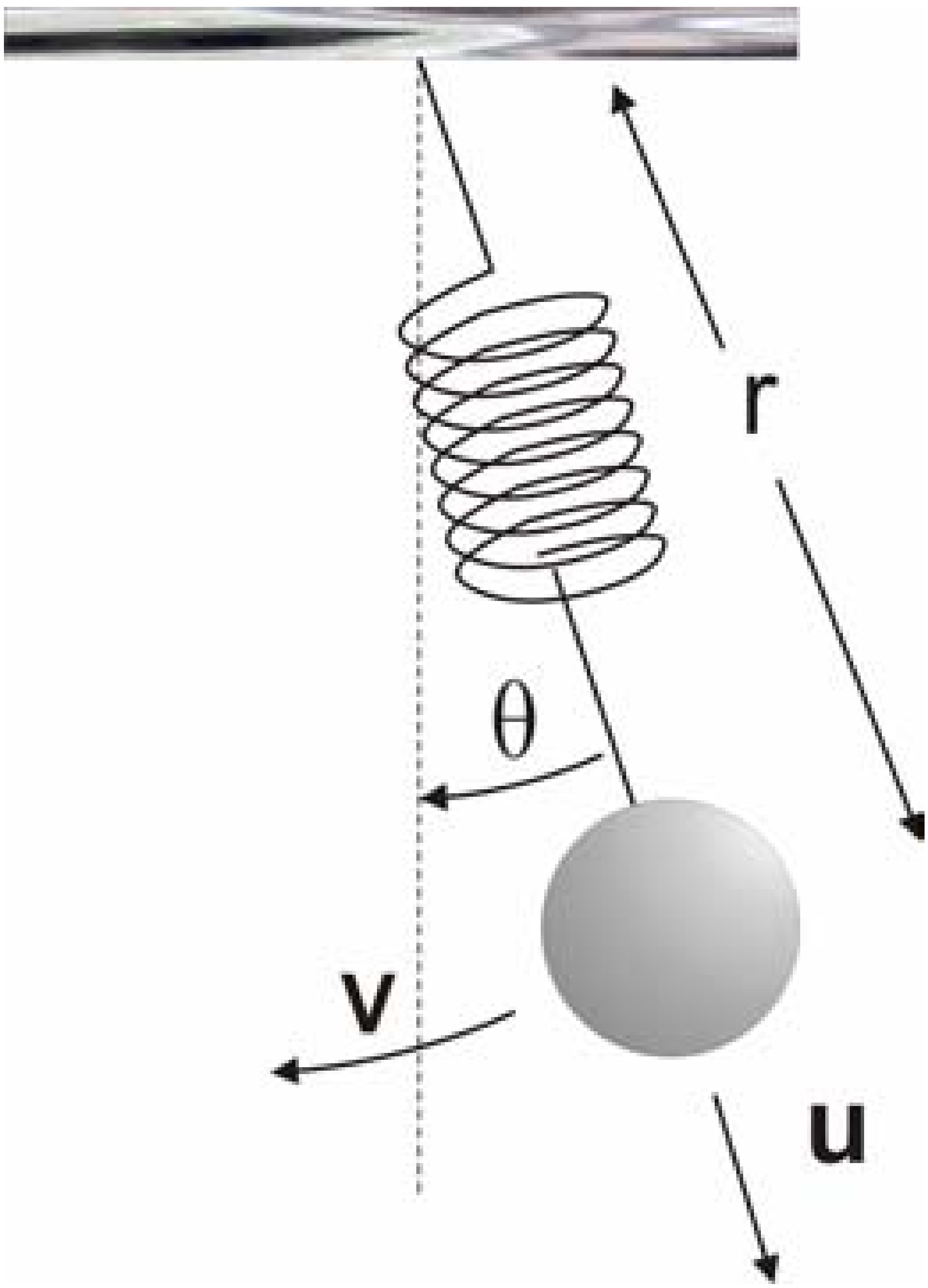
$$\frac{du}{dt} = \frac{v^2}{r} + g(\cos\theta - 1) - K(r - r_0) - \gamma u, \tag{5}$$

$$\frac{dr}{dt} = u, \tag{6}$$

where the velocities u and v and the coordinates r and θ are as indicated in the figure, K is the spring constant, γ is a damping coefficient and r_0 is the radius of the pendulum when at rest.

Part 1

Assume that the amplitude of the motion is small so that $r' = |r - r_0| \ll 1$ and $\theta \ll 1$. Derive the linearized equations for r', $u' = dr'/dt$, θ, and $v = d\theta/dt$. Show that in this limit the radial and azimuthal motions are independent of each other. Solve the equations for r' and θ and show that they represent two fundamental types of oscillations: a "swinging" mode and a "bouncing" mode.


r
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Part 2
Derive the energy equation for the pendulum, i.e. multiply the u-momentum equation by u and the v-momentum equation by v to obtain an equation of the form,

$$\frac{d}{dt}\text{Energy} = -\mu u^2. \tag{7}$$

Show that the energy is composed of two parts, one associated with the swinging mode, the other associated with the bouncing mode.

Part 3
The swing and bouncing oscillations are coupled through nonlinear terms. These oscillations affect each other in various ways. By changing the radius, the bouncing mode affects the moment arm for the swinging mode, while the centrifugal accelerations associated with the swinging mode contribute to radial motion.

Suppose we are only interested in the swinging mode. To what extent can we ignore the bouncing mode? Is there radial motion that we associate with the swinging mode? Physically, we would expect that the swinging mode is decreasingly coupled to the bouncing mode as we make the spring constant, K, ever larger. In order to focus on the evolution of the swinging mode, it is convenient to normalize all length scales by r_0 , all velocities by $\sqrt{gr_0}$, and time by the swinging mode period $\sqrt{r_0/g}$. Show that the nondimesional equations take the form,

$$\frac{dv}{dt} = -\frac{uv}{r} - \sin\theta, \tag{8}$$

$$\frac{d\theta}{dt} = \frac{v}{r}, \tag{9}$$

$$\frac{du}{dt} = \frac{v^2}{r} + \cos\theta - 1 - \nu(r-1) - \mu u, \tag{10}$$

$$\frac{dr}{dt} = u, \tag{11}$$

and express μ and ν in terms of the various dimensional parameters of the problem.

In the limit of large ν, we would expect that $r = 1$ and $u = 0$ would be a good approximation. In that case, (8) and (10) decouple from the other equations and can be solved analytically. This suggests that in the limit of large ν we might find solutions of the form,

$$r = 1 + \frac{r_1}{\nu} + O\left(\frac{1}{\nu^2}\right), \tag{12}$$

$$u = 0 + \frac{u_1}{\nu} + O\left(\frac{1}{\nu^2}\right), \tag{13}$$

$$v = v_0 + \frac{v_1}{\nu} + O\left(\frac{1}{\nu^2}\right), \tag{14}$$

$$\theta = \theta_0 + \frac{\theta_1}{\nu} + O\left(\frac{1}{\nu^2}\right). \tag{15}$$

Find expressions for r_1 and u_1 in terms of the order zero variables, by substituting these expansions in ν into the governing equations and matching like powers of $1/\nu$. These expressions should not contain time derivatives, i.e. they should represent a balance condition.