THE JOURNAL OF FINANCE • VOL. LXI, NO. 6 • DECEMBER 2006

Trading Volume: Implications of an Intertemporal Capital Asset Pricing Model

ANDREW W. LO and JIANG WANG*

ABSTRACT

We derive an intertemporal asset pricing model and explore its implications for trading volume and asset returns. We show that investors trade in only two portfolios: the market portfolio, and a hedging portfolio that is used to hedge the risk of changing market conditions. We empirically identify the hedging portfolio using weekly volume and returns data for U.S. stocks, and then test two of its properties implied by the theory: Its return should be an additional risk factor in explaining the cross section of asset returns, and should also be the best predictor of future market returns.

FUNDAMENTAL SHOCKS TO THE ECONOMY DRIVE BOTH THE SUPPLY and demand of financial assets and their prices. Therefore, any asset pricing model that attempts to establish a structural link between asset prices and underlying economic factors also establishes links between prices and quantities such as trading volume since economic fundamentals such as the investors' preferences and the assets' future payoffs determine the joint behavior of prices and quantities.[1] The construction and empirical implementation of any asset pricing model should therefore involve both price and quantity as key elements. Indeed, from a purely empirical perspective, the joint behavior of price and quantity reveals more information about the relation between asset prices and economic factors than do prices alone. The asset pricing literature, however, has focused more on prices and much less on quantities. For example, empirical investigations of well known asset pricing models such as the Capital Asset Pricing Model (CAPM) and its intertemporal (ICAPM) extensions focus exclusively on prices and returns, completely ignoring the information contained in quantities. In

*Lo is from MIT Sloan School of Management, and National Bureau of Economic Research (NBER), and Wang is from MIT Sloan School of Management, NBER, and China Center for Financial Research. The authors thank Joon Chae, Ilan Guedj, Jannette Papastaikoudi, Antti Petajisto, and Jean-Paul Sursock for excellent research assistance, and Jonathan Lewellen for providing his industry classification scheme. They are grateful to Wayne Ferson and seminar and conference participants at Boston College, the Chinese University of Hong Kong, Duke University, Georgetown University, Harvard Business School, London Business School, New York University, Oxford University, the Shenzhen Stock Exchange, UCLA, University of North Carolina, University of Pennsylvania, University of Toronto, the 8th World Congress of the Econometric Society, the 2001 Lectures in Financial Economics in Beijing and Taipei, the 2005 CORE Lectures at the Université Catholique de Louvain for helpful comments and suggestions. Financial support from the MIT Laboratory for Financial Engineering and the National Science Foundation (Grant No. SBR–9709976) is gratefully acknowledged.

[1] See, for example, Campbell, Grossman, and Wang (1993) and Wang (1994).

this paper, we hope to show that even if our main interest is the behavior of prices, valuable information about price dynamics can be extracted from trading volume.

We begin by developing an intertemporal capital asset pricing model of multiple assets in the spirit of Merton's ICAPM (Merton, 1973). In our model, assets are exposed to market risk and the risk of changes in market conditions.[2] As a result, investors wish to hold two distinct portfolios of risky assets, namely, the market portfolio and a hedging portfolio. The market portfolio allows investors to adjust their exposure to market risk, and the hedging portfolio allows them to hedge the risk of changes in market conditions. In equilibrium, a two-factor linear pricing model holds, where the two factors are the returns on the market portfolio and the hedging portfolio, respectively.

We then explore the implications of this model on the joint behavior of volume and returns. Since investors hold only two portfolios, trading volume also exhibits a two-factor structure. The first factor arises from trades in the market portfolio and the second from trades in the hedging portfolio. More importantly, the factor loading of each asset's trading volume on the hedging portfolio factor is proportional to that asset's portfolio weight in the hedging portfolio. This remarkable property of the trading volume of individual assets suggests a way to identify the hedging portfolio from a rather unexpected source: volume data.

Using weekly returns and trading volume for NYSE and AMEX stocks from 1962 to 2004, we implement the model empirically. From the trading volume of individual stocks, we construct the hedging portfolio and its returns. We find that the hedging portfolio return consistently outperforms other predictors in forecasting future returns to the market portfolio. We then use the returns to the hedging and market portfolios as two risk-factors in a cross-sectional test along the lines of Fama and MacBeth (1973), and find that the hedging portfolio is comparable to other factors in explaining the cross-sectional variation of expected returns. Collectively, these results provide concrete economic foundations for determining risk factors beyond the market portfolio for dynamic equilibrium asset pricing models.

In Section I, we present our intertemporal equilibrium model of asset-pricing and trading volume. In Section II, we explore the model's implications for volume and returns. Section III contains a description of the data used in our empirical implementation of the model, as well as an outline of the construction of the hedging portfolio. In Section IV, we compare the forecast power of the hedging portfolio with other factors, and we perform cross-sectional tests of the hedging portfolio as a risk factor in Section V. We conclude in Section VI.

I. The Model

In this section, we develop an intertemporal equilibrium asset pricing model with multiple assets and heterogeneous investors. Since our purpose is to derive

[2] One example of changes in market conditions is that of changes in the investment opportunity set considered by Merton (1971, 1973).

qualitative implications for the joint behavior of return and volume, the model in subsection A is kept as parsimonious as possible. We discuss several generalizations of the model in subsection B.

A. The Economy

We consider an economy defined on a set of discrete dates $t = 0, 1, 2, \ldots$. There are J risky assets, or "stocks," in the economy. Let D_{jt} denote the date t dividend for each share of stock j, $j = 1, \ldots, J$, and $D_t \equiv (D_{1t}; \ldots; D_{Jt})$ denote the column vector of dividends.[3] Without loss of generality, we normalize the number of shares outstanding for each stock to be one.

A stock portfolio can be expressed in terms of its shares of each stock, denoted by $S \equiv (S_1; \ldots; S_J)$, where S_j is the number of stock-j shares in the portfolio $(j = 1, \ldots, J)$. A portfolio of particular importance is the market portfolio, denoted by S_M, which is given by

$$S_M = (1; \ldots; 1) \tag{1}$$

under our normalization. The dividend on the market portfolio is then given by $D_{Mt} \equiv S_M' D_t$, which is the aggregate dividend.

In addition to stocks, there is also a risk-free bond that yields a constant, positive rate of interest, r, per time period.

There are I investors in the economy, each endowed with equal shares of the stocks and no bonds. Every period, investor i, $i = 1, \ldots, I$, maximizes his expected utility of the following form:

$$\mathrm{E_t}\left[-e^{-W_{t+1}^i - \left(\lambda_X X_t + \lambda_Y Y_t^i\right) D_{Mt+1} - \lambda_Z \left(1 + Z_t^i\right) X_{t+1}}\right], \tag{2}$$

where W_{t+1}^i is investor i's next-period wealth, X_t, Y_t^i, and Z_t^i are three one-dimensional state variables, and λ_X, λ_Y, and λ_Z are nonnegative constants. The utility function specified in (2) is state-dependent.[4] We further assume

$$\sum_{i=1}^{I} Y_t^i = \sum_{i=1}^{I} Z_t^i = 0, \tag{3}$$

where $t = 0, 1, \ldots$.

[3] Throughout this paper, we adopt the following convention: For a set of elements $e_1, \ldots, e_n$, $(e_1; \ldots; e_n)$ denotes the column vector and $(e_1, \ldots, e_n)$ denotes the row vector from these elements.

[4] Using a multiasset extension of the setting in Wang (1994)—which is similar to our setting here except that, in addition to traded assets, investors also hold nontraded assets whose payoffs are driven by exogenous state variables, denoted by X_t and Y_t^i—we can show that the resulting value function takes the form that is qualitatively identical to the utility function given in (2). For brevity, we directly assume the form of the (indirect) utility function here because our purpose is to capture the trading behavior of investors under the given risk structure.

For simplicity, we assume that all the exogenous shocks—$D_t, X_t, \{Y_t^i, Z_t^i, i = 1, \ldots, I\}$—are independently and identically distributed (I.I.D.) over time with zero means. For tractability, we further assume that D_t and X_t are jointly normally distributed:

$$u_t \equiv \begin{pmatrix} D_t \\ X_t \end{pmatrix} \overset{d}{\sim} N(\cdot, \sigma), \quad \text{where} \quad \sigma = \begin{pmatrix} \sigma_{DD} & \sigma_{DX} \\ \sigma_{XD} & \sigma_{XX} \end{pmatrix}. \tag{4}$$

Without loss of generality, σ_{DD} is assumed to be positive definite.

B. Discussion

Our model has several features that might seem unusual. Most importantly, the specification in (2) assumes investors have a myopic but state-dependent utility function. We use this utility function to capture the dynamic nature of the investment problem without having to explicitly solve a dynamic optimization problem. This utility function should be interpreted as the equivalent of a value function from an appropriately specified dynamic optimization problem (see, e.g., Wang (1994) and Lo and Wang (2003)). In particular, it is possible to specify a canonical dynamic optimization problem for investors in which they have state-independent utility over their lifetime consumption such that the resulting value function—a function of wealth and the state variables—has a form similar to the state-dependent utility function in (2).[5] Therefore, for expositional simplicity, we start with (2).

The state dependence of the utility function has the following properties. The marginal utility of wealth depends on the dividend of the market portfolio (the aggregate dividend), as reflected in the second term in the exponential of the utility function. When the aggregate dividend increases, the marginal utility of wealth declines. There are many ways to motivate this type of utility function. For example, utility can be derived from wealth relative to the market, rather than the level of wealth itself (see, e.g., Abel (1990), Campbell and Cochrane (1999)). Alternatively, in addition to their stock investments, investors can be exposed to other risks that are correlated with the market (see, e.g., Wang (1994)). The marginal utility of wealth also depends on future state variables, and in particular X_{t+1}, as reflected in the third term in the exponential of the utility function. The motivation for allowing such dependence is as follows. Since the state variables determine stock returns in equilibrium, the value function (or indirect utility function) of an investor who optimizes dynamically would depend on these state variables. Without modeling the dynamic optimization problem explicitly, we impose such dependence on the (myopic) utility function. This dependence introduces dynamic hedging motives in investors' portfolio choices (see Merton (1971) for a discussion of dynamic hedging).

[5] As Wang (1994) shows, under CARA and time-separable preferences over lifetime consumption and Gaussian processes for asset returns and income, an investor's value function has a similar form to (2).

Another simplification in the model is the I.I.D. assumption for the state variables. This might leave the impression that the model is effectively static. However, this is not the case since the state-dependence of the utility function introduces important dynamics over time, allowing for richer state-variable dynamics in particular without changing the main properties of the model.

The particular form of the utility function and normality of the state variables are assumed for tractability, and such assumptions are restrictive. We hope that the qualitative predictions of our model are not sensitive to these assumptions.

We also assume an exogenous interest rate for the bond, and do not require the bond market to clear. This modeling choice allows us to simplify our analysis and focus squarely on the stock market. It will become clear later that changes in the interest rate are not important for the issues we consider in this paper. Moreover, from an empirical point of view, at the frequency of interest here (weekly), changes in interest rates are usually small.

C. Equilibrium

Let $P_t \equiv (P_{1t}; \ldots ; P_{Jt})$ and $S_t^i \equiv (S_{1t}^i; \ldots ; S_{Jt}^i)$ be the column vectors of ex-dividend stock prices and investor i's stock holdings, respectively. We now derive the equilibrium of the economy.

DEFINITION 1: *An equilibrium is given by a price process* $\{P_t : t = 0, 1, \ldots\}$ *and the investors' stock positions* $\{S_t^i : i = 1, \ldots , I; t = 0, 1, \ldots\}$ *such that:*

1. S_t^i *solves investor i's optimization problem*

$$S_t^i = \arg\ \max \mathrm{E}\left[-e^{-W_{t+1}^i - \left(\lambda_X X_t + \lambda_Y Y_t^i\right) D_{Mt+1} - \lambda_Z \left(1 + Z_t^i\right) X_{t+1}}\right] \tag{5}$$

$$\text{s. t.} \quad W_{t+1}^i = W_t^i + S_t^{i\prime}[D_{t+1} + P_{t+1} - (1+r)P_t]. \tag{6}$$

2. *The stock market clears:*

$$\sum_{i=1}^{i} S_t^i = S_M. \tag{7}$$

This definition of equilibrium is standard except that here the bond market does not clear. As we discussed earlier, the interest rate is given exogenously and there is an infinitely elastic supply of bonds at that rate.

For $t = 0, 1, \ldots,$ let Q_{t+1} denote the vector of excess dollar returns on the stock, that is,

$$Q_{t+1} \equiv D_{t+1} + P_{t+1} - (1+r)P_t. \tag{8}$$

Thus, $Q_{jt+1} = D_{jt+1} + P_{jt+1} - (1 + r)P_{jt}$ gives the dollar return on one share of stock j in excess of its financing cost for period $t + 1$. For the remainder of the paper, we simply refer to Q_{jt+1} as the dollar return of stock j, omitting the qualifier "excess." The dollar return Q_{jt+1} differs from the conventional return measure R_{jt+1}, which is the dollar return normalized by the share price: $R_{jt+1} \equiv Q_{jt+1}/P_{jt}$. We refer to R_{jt+1} simply as the return on stock j in period $t+1$.

We can now state the solution to the equilibrium in the following theorem:

THEOREM 1: *The economy defined above has a unique linear equilibrium in which*

$$P_t = -a - bX_t \tag{9}$$

and

$$S_t^i = \left(1/I - \lambda_Y Y_t^i\right)S_M - \left[\lambda_Y (b'\iota) Y_t^i + \lambda_Z Z_t^i\right](\sigma_{QQ})^{-1}\sigma_{QX}, \tag{10}$$

where ι is a vector of ones,

$$\sigma_{QQ} = \sigma_{DD} - (b\,\sigma_{XD} + \sigma_{DX} b') + \sigma_X^2 b\, b', \tag{11a}$$

$$\sigma_{QX} = \sigma_{DX} - \sigma_X^2 b, \tag{11b}$$

$$a = \frac{1}{r}\left(\bar{\alpha}\sigma_{QQ} S_M + \lambda_Z \sigma_{QX}\right), \tag{11c}$$

$$b = \lambda_X \left[(1+r) + (\lambda_Z \sigma_{XD} S_M)\right]^{-1} \sigma_{DD} S_M, \tag{11d}$$

and $\bar{\alpha} = 1/I$.

The nature of the equilibrium is intuitive. In our model, an investor's utility function depends not only on his wealth, but also on the stock payoffs directly. In other words, even if he holds no stocks, his utility fluctuates with the stocks' payoffs. Such "market spirits "—as opposed to "animal spirits "—affect his demand for stocks, in addition to the usual factors such as the stocks' expected returns. We measure the market spirits of investor i by $(\lambda_X X_t + \lambda_Y Y_t^i)$. If $(\lambda_X X_t + \lambda_Z Y_t^i)$ is positive, investor i extracts positive utility when the aggregate stock payoff is high. Such a positive "attachment " to the market makes holding stocks less attractive to him. If $(\lambda_X X_t + \lambda_Y Y_t^i)$ is negative, he has a negative attachment to the market, which makes holding stocks more attractive. At the aggregate level, such market spirits, which are captured by X_t, affect aggregate stock demand, which in turn affect equilibrium stock prices. Given the particular form of the utility function, X_t affects equilibrium stock prices linearly. In contrast, idiosyncratic differences among investors in the magnitudes of their market spirits, which are captured by Y_t^i, offset each other at the aggregate level, and thus they do not affect equilibrium stock prices. However, they do affect individual investors' stock holdings. As the first term of (10) shows, investors with positive Y_t^is hold less stocks (that is, they gain utility by merely observing stock payoffs).

Since the aggregate utility variable X_t drives stock prices, it also drives stock returns. In fact, the expected return of stocks changes with X_t (see the discussion in the next section). The form of the utility function also implies that the investors' utility functions depend directly on X_t, which fully characterizes the market conditions that investors face, that is, the investment opportunities. Such a dependence arises endogenously when investors optimize dynamically. In our setting, however, we assume that investors optimize myopically, so we insert such a dependence directly into the utility function. This dependence induces investors to care about future market conditions when choosing their portfolios. In particular, they prefer those portfolios whose returns can help them to smooth fluctuations in their utility that are due to changes in market conditions. Such a preference gives rise to the hedging component in their asset demand, which is captured by the second term in (10).

II. The Behavior of Returns and Volume

Given the intertemporal CAPM defined above, we can derive its implications for the behavior of returns and volume. For stocks, their dollar return vector can be reexpressed as

$$Q_{t+1} = ra + (1+r)bX_t + \tilde{Q}_{t+1}, \tag{12}$$

where $\tilde{Q}_{t+1} \equiv D_{t+1} - bX_{t+1}$ denotes the vector of unexpected dollar returns on the stocks, which are I.I.D. over time with zero mean. Equation (12) shows that the expected returns on stocks change over time. In particular, they are driven by a single state variable, X_t.

The investors' stock holdings can be expressed in the following form:

$$S_t^i = h_{Mt}^i S_M + h_{Ht}^i S_H \quad \forall\, i = 1, 2, \ldots, I, \tag{13}$$

where $h_{Mt}^i \equiv (1/I) - \lambda_Y Y_t^i$, $h_{Ht}^i \equiv \lambda_Y (b' S_M) Y_t^i - \lambda_Z Z_t^i$, and

$$S_H \equiv \left(\sigma_{QQ}\right)^{-1} \sigma_{QX}. \tag{14}$$

Equation (13) states that two-fund separation holds for the investors' stock investments. That is, the stock investments of all investors can be viewed as investments in two common funds: the market portfolio S_M and the hedging portfolio S_H.[6] In our model, these two portfolios, expressed in terms of stock shares, are constant over time.

The particular structure of the returns and the investors' portfolios lead to several interesting predictions about the behavior of volume and returns, which we summarize in Propositions 1–4.

[6] Note that the investors' total portfolios satisfy *three-fund* monetary separation—the risk-free bond and the two stock funds. For our discussion here, we restrict our attention to investors' stock investments and always focus on the two stock funds.

A. The Cross-Section of Volume

Given heterogeneity in preferences, which change over time, investors trade among themselves to achieve their optimal stock holdings. The volume of trade can be measured by the turnover ratio (see Lo and Wang (2000)). Since we normalize the total number of shares outstanding to be one for all stocks, the turnover of a stock, say, stock j, is given by

$$\tau_{jt} \equiv \frac{1}{2}\sum_{i=1}^{I} \left| \left(h^i_{Mt} - h^i_{Mt-1}\right) + \left(h^i_{Ht} - h^i_{Ht-1}\right) S_{Hj} \right|, \quad \forall j = 1, \ldots, J. \tag{15}$$

Let τ_t denote the vector of turnover for all stocks. We have the following proposition for the cross-section of volume:

PROPOSITION 1: *When trading in the hedging portfolio is small relative to trading in the market portfolio,*[7] *the two-fund separation in investors' stock holdings leads to a two-factor structure for stock turnover*

$$\tau_t \approx S_M F_{Mt} + S_H F_{Ht}, \tag{16}$$

where

$$F_{Mt} = \frac{1}{2}\sum_{i=1}^{I} \left|h^i_{Mt} - h^i_{Mt-1}\right| \quad \textit{and} \quad F_{Ht} = \frac{1}{2}\sum_{i=1}^{I} \left(h^i_{Ht} - h^i_{Ht-1}\right)\mathrm{Sign}\left(h^i_{Mt} - h^i_{Mt-1}\right). \tag{17}$$

In the special case where one-fund separation holds for stock holdings (when $X_t = 0$, $\forall t$), turnover has an exact one-factor structure, $\tau_t = S_M F_{Mt}$. Moreover, the loadings of individual turnover on the common factor are identical across all stocks, hence turnover is identical across all stocks. This is not surprising—in the case of one-fund separation for stock investments, investors trade in one stock portfolio, which has to be the market portfolio, and thus they must trade all the stocks in the same proportions (in shares). Consequently, the turnover must be the same for all stocks.[8]

In the general case where two-fund separation holds for stock investments, and assuming that the trading in the market portfolio dominates the trading in the hedging portfolio, turnover has a two-factor structure (after a linear approximation) as given in (16). Although parameter restrictions can be imposed in the theoretical model to satisfy the condition that trading in the hedging portfolio be small (see the Appendix), it is an empirical question as to whether this and the two-factor model are plausible, which we address in the empirical analysis of Section III.

It is important to note that the loading of stock j's turnover on the second factor is proportional to its share weight in the hedging portfolio. Therefore,

[7] This imposes certain restrictions on the λ coefficients and the other model parameters. See the proof in the Appendix for more details.

[8] For a discussion on the implications of mutual fund separation on the cross-sectional behavior of volume, see Lo and Wang (2000). See also Tkac (1996).

if we can empirically identify the two common factors F_{Mt} and F_{Ht}, the stocks' loadings on the second factor will allow us to identify the hedging portfolio. In our empirical analysis, we explore the information that the cross section of volume contains. As we discuss below, the hedging portfolio has important properties that allow us to better understand the behavior of returns. Merton (1971, 1973) considers the properties of hedging portfolios in a continuous-time framework as a characterization of equilibrium. Our discussion here follows Merton in spirit, but is set in a discrete-time, equilibrium environment.

B. Time Series Implications for the Hedging Portfolio

By the definition of the hedging portfolio in (14), it is easy to show that its current return gives the best forecast of future market returns.

Let Q_{Mt+1} denote the dollar return on the market portfolio in period $t+1$, and Q_{Ht+1} denote the dollar return on the hedging portfolio. Then,

$$Q_{Mt+1} = S'_M Q_{t+1} \quad \text{and} \quad Q_{Ht+1} = S'_H Q_{t+1}. \tag{18}$$

For an arbitrary portfolio S, its dollar return in period t, which is $Q_{St} \equiv S'Q_t$, can serve as a predictor for the next-period dollar return of the market:

$$Q_{Mt+1} = \delta_0 + \delta_1 Q_{St} + \varepsilon_{Mt+1}. \tag{19}$$

The predictive power of S is measured by the R^2 of the above regression. We can solve for the portfolio that maximizes the R^2. The solution, up to a scaling constant, is the hedging portfolio. Thus, we have the following result:

PROPOSITION 2: *Among the returns of all portfolios, the dollar return of the hedging portfolio, S_H, provides the best forecast for the future dollar return of the market.*

In other words, if we regress the market dollar return on the lagged dollar-return of any set of portfolios, the hedging portfolio must yield the highest R^2.

C. Cross-Sectional Implications for the Hedging Portfolio

We now turn to the predictions of our model for the cross section of returns. Let Q_{pt+1} denote the dollar return of a stock portfolio, $\bar{Q}_{pt+1} \equiv \mathrm{E_t}[Q_{pt+1}]$ its conditional expectation at time t, $\bar{Q}_p$ its unconditional expectation, and $\tilde{Q}_{pt+1} \equiv Q_{pt+1} - \bar{Q}_{pt+1}$ its unexpected dollar return in period $t+1$. Then, $\tilde{Q}_{Mt+1}$ and $\tilde{Q}_{Ht+1}$ denote the unexpected dollar returns on the market portfolio and the hedging portfolio, respectively, and

$$\sigma_M^2 \equiv \mathrm{Var}[\tilde{Q}_{Mt+1}], \;\; \sigma_H^2 \equiv \mathrm{Var}[\tilde{Q}_{Ht+1}], \;\; \text{and } \sigma_{MH} \equiv \mathrm{Cov}[\tilde{Q}_{Mt+1}, \tilde{Q}_{Ht+1}] \tag{20}$$

denote their conditional variances and covariances. From Theorem 1, we have

$$\bar{Q} = \bar{\alpha}\sigma_{QQ}\iota + \lambda_Z \sigma_{QX} \tag{21a}$$

$$\bar{Q}_M = \bar{\alpha}\sigma_M^2 + \lambda_Z \sigma_{MH} \tag{21b}$$

$$\bar{Q}_H = \bar{\alpha}\sigma_{MH} + \lambda_Z \sigma_H^2, \tag{21c}$$

where $\sigma_M^2 = S_M' \sigma_{QQ} S_M$, $\sigma_H^2 = \sigma_{XQ}(\sigma_{QQ})^{-1}\sigma_{QX}$, $\sigma_{MH} = \iota'\sigma_{QX}$, σ_{QQ}, and σ_{QX} are given in Theorem 1. Equation (21) characterizes the cross-sectional variation in the stocks' expected dollar returns.

To develop more intuition about (21), we first consider the special case in which $X_t = 0, \forall\, t$. In this case, returns are I.I.D. over time. The risk of a stock is measured by its covariability with the market portfolio. We have the following result:

PROPOSITION 3: *When $X_t = 0, \forall\, t$, we have*

$$E[\tilde{Q}_{t+1} | \tilde{Q}_{Mt+1}] = \beta_M \tilde{Q}_{Mt+1}, \tag{22}$$

where

$$\beta_M \equiv \text{Cov}[\tilde{Q}_{t+1}, \tilde{Q}_{Mt+1}]/\text{Var}[\tilde{Q}_{Mt+1}] = \sigma_{DD}\iota/(\iota'\sigma_{DD}\iota) \tag{23}$$

is the vector of the stocks' market betas. Moreover,

$$\bar{Q} = \beta_M \bar{Q}_M, \tag{24}$$

where $\bar{Q}_M = \bar{\alpha}\sigma_M^2 \geq 0$.

Obviously in this case, the CAPM holds for dollar returns; it can be shown that it also holds for returns.

In the general case where X_t changes over time, there is additional risk due to changing market conditions (dynamic risk). Moreover, this risk is represented by the dollar return of the hedging portfolio, which is denoted by $Q_{Ht} \equiv S_H' Q_t$. In this case, the risk of a stock is measured by its risk with respect to the market portfolio *and* its risk with respect to the hedging portfolio. In other words, there are two risk-factors, the (contemporaneous) market risk and the (dynamic) risk of changing market conditions. The expected returns of stocks are then determined by their exposures to these two risks and the associated risk premia. The result is summarized in the following proposition:

PROPOSITION 4: *When X_t changes over time, we have*

$$\text{E}[\tilde{Q}_{t+1} \,|\, \tilde{Q}_{Mt+1}, \tilde{Q}_{Ht+1}] = \beta_M \tilde{Q}_{Mt+1} + \beta_H \tilde{Q}_{Ht+1}, \tag{25}$$

where

$$(\beta_M, \beta_H) = \text{Cov}[\tilde{Q}_{t+1}, (\tilde{Q}_{Mt+1}, \tilde{Q}_{Ht+1})]\{\text{Var}[(\tilde{Q}_{Mt+1}, \tilde{Q}_{Ht+1})]\}^{-1} \tag{26}$$

$$= (\sigma_{QM}, \sigma_{QH}) \begin{pmatrix} \sigma_M^2 & \sigma_{MH} \\ \sigma_{MH} & \sigma_H^2 \end{pmatrix}^{-1} \tag{27}$$

is the vector of the stocks' market betas and hedging betas. Moreover, the stocks' expected dollar returns satisfy

$$\bar{Q} = \beta_M \bar{Q}_M + \beta_H \bar{Q}_H, \tag{28}$$

where

$$\bar{Q}_M = \bar{\alpha}\sigma_M^2 + \lambda_Z \sigma_{MH}$$

and

$$\bar{Q}_H = \bar{\alpha}\sigma_{MH} + \lambda_Z \sigma_H^2.$$

Therefore, a stock's risk is measured by its beta with respect to the market portfolio and its beta with respect to the hedging portfolio. The expected dollar-return on the market portfolio is the market risk premium and the expected dollar-return on the hedging portfolio is the dynamic risk premium. Equation (28) states that the premium on a stock is then given by the sum of the product of its exposure to each risk with the corresponding risk premium.

Under constant market conditions ($X_t = 0, \forall t$), the market risk premium, $\bar{Q}$, is always positive. However, under changing market conditions, the market risk premium need not be positive. In particular, when σ_{MH} is significantly negative (λ_Z is assumed to be positive), $\bar{Q}$ can be negative. This is simply because the premium is determined by the covariance between the market return and investors' marginal utility, which depends on both their wealth and the other state variables. In particular, the positive covariance between market returns and investors' wealth yields a positive premium to the market portfolio, whereas the negative covariance between market returns and the state variable X_t that drives the utility function yields a negative premium. The total premium on the market portfolio is the sum of these two components, which can be negative if the second component dominates.

The pricing relation we obtain in Proposition 4 is in the spirit of Merton's Intertemporal CAPM in a continuous-time framework (Merton (1971)). However, it is important to note that Merton's result is a characterization of the pricing relation under a class of proposed price processes, and no equilibrium is provided to support these price processes. In contrast, our pricing relation is derived from a dynamic equilibrium model—in this sense, our model provides a specific equilibrium model for which Merton's characterization holds.

If we can identify the hedging portfolio empirically, its return should provide the second risk-factor. Cross-sectional differences among the stocks' expected returns can then be fully explained by their exposures to the two risks—market risk and dynamic risk—and can be measured by their market and hedging betas.

D. Further Discussion

We have derived the joint behavior of returns and volume, both in the cross section and over time, from a specific intertemporal CAPM in which time variation in the total risk tolerance of the economy causes the market risk premium to change over time, which gives rise to dynamic risk in addition to static market risk. However, it should be clear from the discussion above that the return/volume implications do not rely on the particular source of dynamic risk. Any model in which dynamic risk is captured by the time variation in the market risk premium will lead to similar return/volume implications. In particular, investors will trade only in two portfolios, the market portfolio and the hedging portfolio, to allocate market and dynamic risks, and the return on the hedging portfolio is an additional risk-factor in explaining the cross section of returns and it also gives the best forecast for future market returns. The actual source of dynamic risk, that is, time variation in the market premium, is not crucial to these results. For example, time variation in the market premium can be easily driven by time-varying market sentiment or, in models more general than that considered here, time variation in the total risk level of the market.

Our version of the intertemporal CAPM can also incorporate other types of dynamic risks. For example, if market liquidity is time-varying, as Chordia, Roll, and Subrahmanyam (2000) and Pastor and Stambaugh (2003) suggest, this can be viewed as dynamic risk. In particular, the pricing implications examined by Pastor and Stambaugh (2003) fall within our ICAPM framework. What is less clear in models based on liquidity considerations is the predictions for volume behavior and the return/volume relation. In these models, volume is used heuristically to empirically gauge the liquidity of an asset. For example, Brennan and Subrahmanyam (1996) use volume as proxy for liquidity and examine how liquidity may help to explain the cross section of asset returns. In this case, liquidity is treated as an asset-specific factor that need not be directly related to risks. Acharya and Pedersen (2002) explicitly model the liquidity cost as correlated with market risk, and then show that even when liquidity-adjusted returns satisfy the CAPM, raw returns do not. In particular, expected returns depend on the market risks of the assets' payoffs as well as their liquidity adjustments.

Despite the common focus on the cross-sectional properties of returns, the approach in liquidity-based models is quite different from ours. The former models rely primarily on empirically motivated models of liquidity or liquidity risk, and test the cross-sectional properties of returns. Our approach is to develop the implications for both volume and returns using an intertemporal equilibrium model, and then test these implications jointly. Although our specification may be less flexible from an empirical perspective, this is due to the fact that we are imposing more structure to guide empirical analysis.

III. An Empirical Implementation

Our empirical analysis of the implications of the model outlined in Sections I and II consists of three parts. In this section, we exploit the model's cross-sectional implications to construct the hedging portfolio from volume data. In Section IV, we examine the ability of the hedging portfolio to forecast future market portfolio returns. Finally in Section V, we investigate the role of the hedging portfolio return as a risk factor in explaining the cross-sectional variation of expected returns.

A. The Data

We use an extract of the CRSP Daily Master File called the "MiniCRSP Returns and Turnover " database described in Lo and Wang (2000) (see also Lim et al. (1998)). This extract consists of weekly return and turnover series for individual stocks traded on NYSE and AMEX from July 1962 to December 2004 (2,217 weeks). The weekly turnover of a stock is simply the sum of its daily turnover, which is the number of shares traded each day normalized by the total number of shares outstanding. We choose weekly periods as a compromise between maximizing the sample size and minimizing the impact of high-frequency return and turnover fluctuations that are likely to be of less direct economic consequence. We also limit our focus to ordinary common shares (CRSP sharecodes 10 and 11 only).

As documented in Lo and Wang (2000) and in many other studies, aggregate turnover seems to be nonstationary, exhibiting a significant time trend and time-varying volatility. For example, the average weekly equal-weighted turnover in the period from 1962 to 1966 is 0.57%, but grows to 2.07% in the period from 1997 to 2004, and the volatilities during these two periods are 0.21% and 0.51%, respectively. Detrending has been advocated by several other authors (e.g., Andersen (1996), Gallant, Rossi, and Tauchen (1992)), and there is no doubt that such procedures may help to induce more desirable time series properties for turnover. However, Lo and Wang (2000) show that the different types of detrending methods, for example, linear, logarithmic, or quadratic, yield detrended time series with markedly different statistical properties. Since we do not have any specific priors or theoretical justification for the kinds of nonstationarities in aggregate turnover, we use the raw data in our empirical analysis. To address the issue of nonstationarities, we conduct our empirical analysis on 5-year subperiods only.[9] For notational convenience, we shall sometimes refer to these subperiods by the following numbering scheme:

Subperiod 1:	July 1962 to December 1966
Subperiod 2:	January 1967 to December 1971
Subperiod 3:	January 1972 to December 1976
Subperiod 4:	January 1977 to December 1981
Subperiod 5:	January 1982 to December 1986
Subperiod 6:	January 1987 to December 1991
Subperiod 7:	January 1992 to December 1996
Subperiod 8:	January 1997 to December 2001
Subperiod 9:	January 2002 to December 2004

[9] Obviously, from a purely statistical perspective, using shorter subperiods does not render a nonstationary time series stationary. However, if the sources of nonstationarity are institutional changes and shifts in general business conditions, confining our attention to shorter time spans does improve the quality of statistical inference. See Lo and Wang (2000) for further discussion.

B. Construction of the Hedging Portfolio

Our first step in empirically implementing the intertemporal model of Sections I and II is to construct the hedging portfolio from turnover data. From (16), we know that in the two-factor model for turnover in Proposition 1, stock j's loading on the second factor F_{Ht} yields the number of shares (as a fraction of its total number of shares outstanding) of stock j in the hedging portfolio. In principle, this identifies the hedging portfolio. However, we face two challenges in practice. First, the two-factor specification (16) is, at best, an approximation for the true data generating process of turnover (even within the context of the model as Proposition 1 states). Second, the two common factors are generally not observable. We address both of these issues in turn.

A more realistic starting point for modeling turnover is the two-factor model

$$\tau_{jt} = F_{Mt} + \theta_{Hj} F_{Ht} + \varepsilon_{jt}, \quad j = 1, \ldots, J, \tag{29}$$

where F_{Mt} and F_{Ht} are the two factors associated with trading in the market portfolio and the hedging portfolio, respectively, θ_{Hj} is the percentage of shares of stock j in the hedging portfolio (as a percentage of its total number of shares outstanding), and ε_{jt} is the error term, which is assumed to be independent across stocks.

Cross-sectional independence of the errors is a restrictive assumption. If, for example, there are other common factors in addition to F_{Mt} and F_{Ht}, then ε_{jt} is likely to be correlated across stocks. The appropriateness of the independence assumption is an empirical matter, and in Lo and Wang (2000), we find evidence supporting a two-factor structure. In particular, the covariance matrices of turnover for a collection of turnover-beta-sorted portfolios generally exhibit two large eigenvalues that dominate the rest. This provides limited justification for assuming that ε_{jt} is independent across stocks.

Since we do not have sufficient theoretical foundation to identify the two common factors F_{Mt} and F_{Ht}, we use two turnover indexes as their proxies, namely, the equal-weighted and share-weighted turnover of the market. Specifically, let N_j denote the total number of trading units, each as a fraction of the total share, for stock j and let $N \equiv \sum_j N_j$ denote the total number of trading units of all stocks.[10] The two turnover indexes are

$$\tau_t^{EW} \equiv \frac{1}{J} \sum_{j=1}^{J} \tau_{jt} = F_{Mt} + n^{EW} F_{Ht} + \varepsilon_t^{EW} \tag{30a}$$

$$\tau_t^{SW} \equiv \sum_{j=1}^{J} \frac{N_j}{N} \tau_{jt} = F_{Mt} + n^{SW} F_{Ht} + \varepsilon_t^{SW}, \tag{30b}$$

[10] By standard terminology, N_j should be the number of shares outstanding for stock j. Given our convention that the total number of shares outstanding is always normalized to one, N_j becomes the number of subunits used in trading stock j.

where

$$n^{EW} = \frac{1}{J}\sum_{j=1}^{J}\theta_{Hj} \quad \text{and} \quad n^{SW} = \sum_{j=1}^{J}\frac{N_j}{N}\theta_{Hj} \tag{31}$$

are the average fraction of trading units of each stock in the hedging portfolio and the fraction of all trading units (of all stocks) in the hedging portfolio, respectively, and ε_t^{EW} and ε_t^{SW} are the error terms for the two indexes.[11] Since the error terms in (29) are assumed to be independent across stocks, the error terms of the two indexes, which are weighted averages of the error terms of individual stocks, become negligible when the number of stocks is large. For the remainder of our analysis, we shall ignore them.

Simple algebra then yields the following relation between individual turnover and the two indexes:

$$\tau_{jt} = \beta_{\tau j}^{SW}\tau_t^{SW} + \beta_{\tau j}^{EW}\tau_t^{EW} + \varepsilon_{jt}, \tag{32}$$

where

$$\beta_{\tau j}^{EW} = \frac{\theta_{Hj} - n^{SW}}{n^{EW} - n^{SW}} \quad \text{and} \quad \beta_{\tau j}^{SW} = \frac{n^{EW} - \theta_{Hj}}{n^{EW} - n^{SW}}. \tag{33}$$

These expressions imply that the following relations for $\beta_{\tau j}^{EW}$ and $\beta_{\tau j}^{SW}$ must hold

$$\beta_{\tau j}^{EW} + \beta_{\tau j}^{SW} = 1, \quad \forall j \tag{34a}$$

$$\frac{1}{J}\sum_{j=1}^{J}\beta_{\tau j}^{EW} = 1. \tag{34b}$$

These relations should come as no surprise since the two-factor specification for turnover, (29), has only J parameters $\{\theta_{Hj}\}$, whereas the transformed two-factor model, (32), has two sets of parameters, $\{\beta_{\tau j}^{EW}\}$ and $\{\beta_{\tau j}^{SW}\}$. The first relation, (34a), exactly reflects the dependence between the parameters and the second relation, (34b), comes from the fact that the coefficients in (32) are independent of the scale of $\{\theta_{Hj}\}$.

Using the MiniCRSP volume database, we can empirically estimate $\{\beta_{\tau j}^{EW}\}$ and $\{\beta_{\tau j}^{SW}\}$ by estimating the following constrained regression:

$$\tau_{jt} = \beta_{\tau j}^{SW}\tau_t^{SW} + \beta_{\tau j}^{EW}\tau_t^{EW} + \varepsilon_{jt}, \quad j = 1, \ldots, J \tag{35a}$$

$$\text{s.t.} \quad \rho\beta_{\tau j}^{EW} + \beta_{\tau j}^{SW} = 1 \tag{35b}$$

$$\sum_{j=1}^{J}\beta_{\tau j}^{EW} = J. \tag{35c}$$

[11] To avoid degeneracy, we need $N_j \neq N_k$ for some $j \neq k$, which is surely valid empirically.

From the estimates $\{\hat{\beta}^{EW}_{\tau j}\}$, we can construct estimates of the portfolio weights of the hedging portfolio in the following manner:

$$\hat{\theta}_{Hj} = (n^{EW} - n^{SW})\hat{\beta}^{EW}_{\tau j} + n^{SW}. \tag{36}$$

However, there are two remaining parameters, n^{EW} and n^{SW}, that need to be estimated. It should be emphasized that these two remaining degrees of freedom are inherent in the model (see equation (29)). When the two common factors are not observed, the parameters $\{\theta_{Hj}\}$ are only identified up to a scaling constant and a rotation. Clearly, equation (29) is invariant when F_{Ht} is rescaled as long as $\{\theta_{Hj}\}$ is also rescaled appropriately. In addition, when the two factors are replaced by their linear combinations, equation (29) remains the same as long as $\{\theta_{Hj}\}$ is also adjusted with an additive constant.[12] Since the hedging portfolio $\{\theta_{Hj}\}$ is defined only up to a scaling constant, we let

$$n^{SW} = 1 \tag{37a}$$

$$n^{EW} - n^{SW} = \phi, \tag{37b}$$

where ϕ is a parameter that we calibrate to the data (see Section IV). This yields the final expression for the J components of the hedging portfolio:

$$\hat{\theta}_{Hj} = \phi\, \hat{\beta}^{EW}_{\tau j} + 1. \tag{38}$$

The normalization $n^{SW} = 1$ sets the total number of shares in the portfolio to a positive value. If $\phi = 0$, the portfolio has an equal percentage of all the shares of each company, implying that it is the market portfolio. Nonzero values of ϕ represent deviations from the market portfolio.

To estimate $\{\beta^{EW}_{\tau j}\}$ and $\{\beta^{SW}_{\tau j}\}$, we first construct the two turnover indexes. Figure 1 plots their time series over the entire sample period from 1962 to 2004. We estimate (35a)–(35b) for each of the seven 5-year subperiods, ignoring the global constraint (35c).[13] Therefore, we estimate both constrained and unconstrained linear regressions of the weekly turnover for each stock on equal- and share-weighted turnover indexes in each of the seven 5-year subperiods of our sample. Table I reports summary statistics for the unconstrained regressions (see Lo and Wang (2005) for the constrained regression results). To provide a clearer sense of the dispersion of these regressions, we

[12] For example, for any a, we have $\forall j$:

$$\tau_{jt} = F_{Mt} + \theta_{Hj}F_{Ht} + \varepsilon_{jt} = (F_{Mt} + aF_{Ht}) + (\theta_{Hj} - a)F_{Ht} + \varepsilon_{jt} = \tilde{F}_{Mt} + \tilde{\theta}_{Hj}F_{Ht} + \varepsilon_{jt}, \tag{47}$$

where $\tilde{F}_{Mt} = F_{Mt} + aF_{Ht}$ and $\tilde{\theta}_{Hj} = \theta_{Hj} - a$.

[13] We ignore this constraint for two reasons. First, given the large number of stocks in our sample, imposing a global constraint such as (35c) requires a prohibitive amount of random access memory in standard regression packages. Second, because of the large number of individual regressions involved, neglecting the reduction of one dimension should not significantly affect any of the final results.

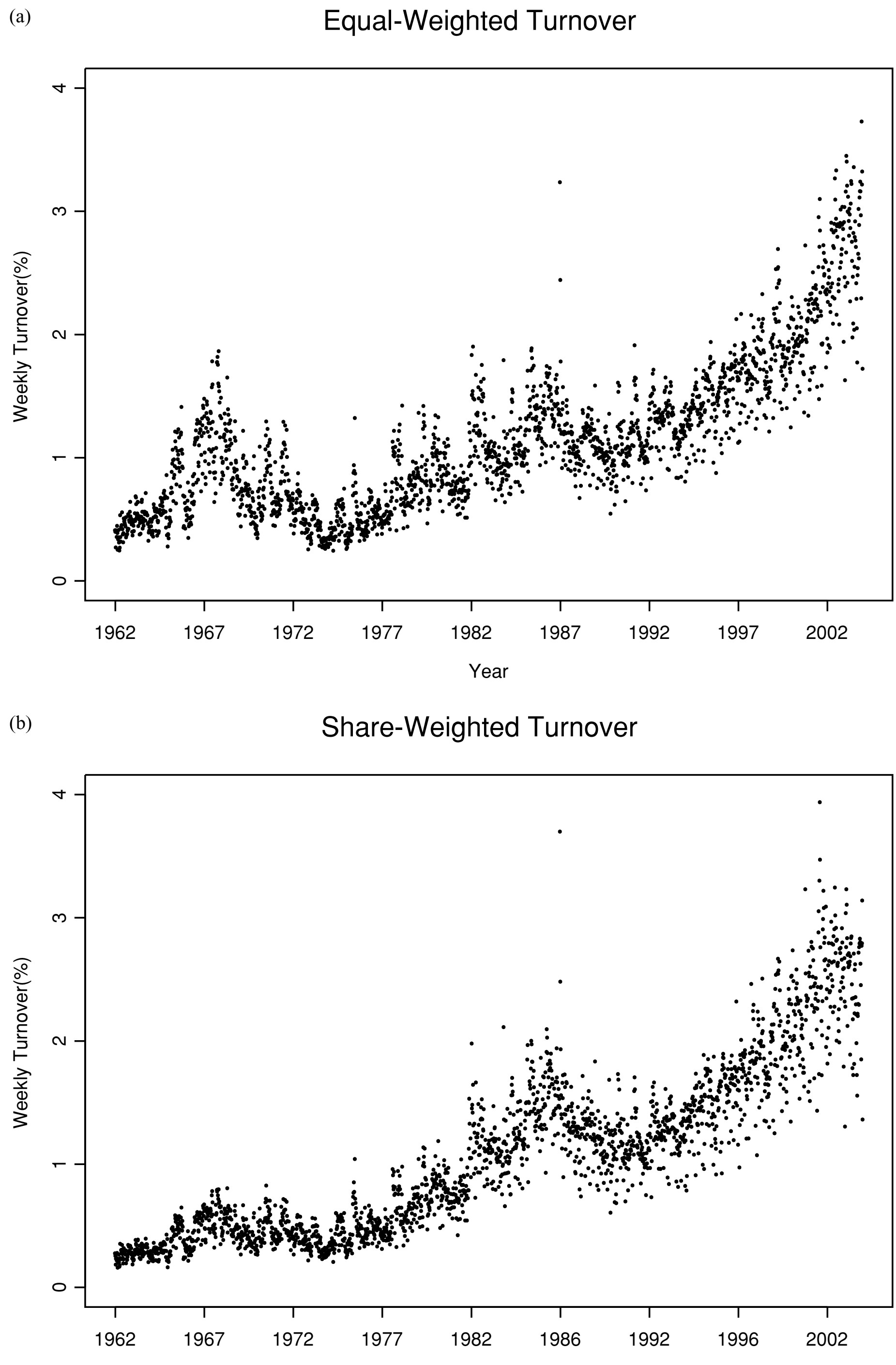

Figure 1. Time series of equal- and share-weighted turnover indices from 1962 to 2004.

Table I

Summary statistics for the unrestricted volume betas using weekly returns and volume data for NYSE and AMEX stocks for three subperiods: January 1967 – December 1971, January 1997 – December 2001, and January 2002 – December 2004. Turnover over individual stocks is regressed on the equal-weighted and share-weighted turnover indices, giving two regression coefficients, β_τ^{EW} and β_τ^{SW}. The stocks are then sorted into 10 deciles by the estimates $\hat{\beta}_\tau^{EW}$, and summary statistics are reported for deciles 1, 4, 7, and 10. The last two columns report the test statistic for the condition that β_τ^{EW} and β_τ^{SW} add up to one.

	Sample	$\hat{\beta}_\tau^{EW}$		$t(\hat{\beta}_\tau^{EW})$		$\hat{\beta}_\tau^{SW}$		$t(\hat{\beta}_\tau^{SW})$		$\overline{R}^2$ (%)		p-value (%)	
Decile	Size	Mean	S.D.	Mean	S.D.	Mean	S.D.	Mean	S.D.	Mean	S.D.	Mean	S.D.
					January 1967 – December 1971 (261 Weeks)								
1	242	−5.11	17.10	−4.31	2.69	12.97	35.91	6.28	3.30	52.1	16.3	1.0	4.3
4	243	−0.18	0.07	−2.30	2.61	1.13	0.73	5.22	3.64	60.6	15.5	16.3	27.9
7	243	1.10	0.20	3.38	1.89	−0.38	0.93	−1.10	1.62	56.0	12.0	20.6	28.1
10	242	7.50	3.60	6.83	2.63	−9.67	5.56	−4.64	2.05	55.6	11.7	10.1	21.8
					January 1997 – December 2001 (261 Weeks)								
1	326	−7.06	21.11	−2.63	1.64	8.37	20.48	3.76	2.25	58.8	19.4	17.5	28.2
4	327	0.09	0.09	0.45	0.69	0.41	0.50	0.74	1.11	53.4	23.1	4.8	17.2
7	327	1.31	0.16	2.28	1.37	−0.48	0.53	−1.20	1.26	54.0	18.3	6.9	19.4
10	326	12.11	32.64	2.95	1.52	−9.41	29.10	−2.28	1.28	46.6	19.0	8.8	20.9
					January 2002 – December 2004 (156 Weeks)								
1	233	−3.80	3.71	−3.32	2.50	5.68	4.75	5.16	3.69	64.2	22.3	13.6	24.9
4	234	0.10	0.07	0.86	0.94	0.32	0.45	1.23	1.83	59.2	24.3	3.8	15.4
7	234	1.06	0.16	3.02	1.90	−0.17	0.73	−1.06	1.89	63.9	21.0	8.3	20.8
10	233	10.46	12.23	4.68	2.93	−8.26	10.94	−3.46	2.35	54.5	22.4	17.0	26.9

first sort them into deciles based on $\{\hat{\beta}_{\tau j}^{EW}\}$, and then compute the means and standard deviations of the estimated coefficients $\{\hat{\beta}_{\tau j}^{EW}\}$ and $\{\hat{\beta}_{\tau j}^{SW}\}$, their t-statistics, and the $\bar{R}^2$s within each decile. To conserve space, we report results for only deciles 1, 4, 7, and 10, and only for subperiods 1, 8, and 9 (see Lo and Wang (2005) for a more complete set of empirical results). The t-statistics indicate that the estimated coefficients are generally significant, even in the fourth and seventh deciles, the average t-statistics for $\{\hat{\beta}_{\tau j}^{EW}\}$ are −2.30 and 3.38, respectively in the first subperiod (of course, we would expect significant t-statistics in the extreme deciles even if the true coefficients were zero, purely from sampling variation). The average $\bar{R}^2$s also look impressive, ranging from 40% to 60% across deciles and subperiods. Clearly, the two-factor model of turnover accounts for a significant amount of variation in the weekly turnover of individual stocks.

Despite the fact that the results in Table I are derived from unconstrained regressions, the constraint seems to be reasonably consistent with the data, with average p-values well above 5% for all but the first decile in the first two subperiods (decile 1 in subperiod 1 has a p-value of 4.0%, and decile 1 in subperiod 2 has a p-value of 4.3%).

IV. The Forecast Power of the Hedging Portfolio

Having constructed the hedging portfolio up to a parameter ϕ, to be determined, we can examine its time-series properties as predicted by the model of Sections I and II. In particular, in this section we focus on the degree to which the hedging portfolio can predict future stock returns, especially the return on the market portfolio. We first construct the returns of the hedging portfolio in Subsection A by calibrating ϕ, and then compare its forecast power with other factors in Subsections B and C.

A. Hedging Portfolio Returns

To construct the return on the hedging portfolio, we begin by calculating its dollar value and dollar returns. Let k denote subperiod $k, k = 2, \ldots, 7$, $V_{jt}(k)$ denote the time-t (end-of-week-t) total market capitalization of stock j in subperiod k, $Q_{jt}(k)$ denote its dividend-adjusted excess dollar return for the same period, $R_{jt}(k)$ denote the dividend-adjusted excess return, and $\theta_j(k)$ denote the estimated share (as a fraction of its total shares outstanding) in the hedging portfolio in subperiod k.

For stock j to be included in the hedging portfolio in subperiod k, which we shall refer to as the "testing period," we require that it has volume data for at least one-third of the sample in the previous subperiod $(k-1)$, which we refer to as the "estimation period." Among the stocks that satisfy this criteria, we eliminate those ranked in the top and bottom 0.5% according to their volume betas (or their share weights in the hedging portfolio) to limit the potential impact of outliers. We let $J_t(k)$ denote the set of stocks that survive these two filters and that have price and return data for week t of subperiod k. The hedging portfolio in week t of subperiod k is then given by

$$\theta_{Hjt}(k) = \begin{cases} \hat{\theta}_{Hj}, & j \in J_t(k) \\ 0, & j \notin J_t(k). \end{cases} \tag{39}$$

The dollar return of the hedging portfolio for week t follows

$$Q_{Ht}(k) \equiv \sum_j \theta_{Hjt}(k) V_{jt-1}(k) R_{jt}(k), \tag{40}$$

and the (rate of) return of the hedging portfolio is given by

$$R_{Ht}(k) \equiv \frac{Q_{Ht}(k)}{V_{Ht-1}(k)}, \tag{41}$$

where

$$V_{Ht-1}(k) \equiv \sum_j \theta_{Hjt}(k) V_{jt-1}(k) \tag{42}$$

is the value of the hedging portfolio at the beginning of the week.

The procedure outlined above yields the return and the dollar return of the hedging portfolio up to the parameter ϕ, which must be calibrated. To do so, we exploit a key property of the hedging portfolio: Its return is the best forecaster of future market returns (see Section II). Therefore, for a given value of ϕ, we can estimate the regression

$$R_{Mt+1} = \delta_0 + \delta_1\{R_{Ht} \text{ or } Q_{Ht}\} + \varepsilon_{Mt+1}, \tag{43}$$

where the single regressor is either the return of the hedging portfolio R_{Ht} or its dollar return for a given choice of ϕ, and then vary ϕ to maximize the $\bar{R}^2$.[14] The $\bar{R}^2$ from the regression of R_{Mt} on the lagged return and dollar return, respectively, of the hedging portfolio varies with the value of ϕ in each of the subperiods, and in all cases, there is a unique global maximum, from which we obtain ϕ. However, for some values of ϕ, the value of the hedging portfolio changes sign, and in these cases, defining the return on the portfolio becomes problematic. Thus, we eliminate these values from consideration, and for all subperiods except subperiods 4 and 7 (i.e., subperiods 2, 3, 5, 6, 8, and 9), the omitted values of ϕ do not consistently affect the choice of ϕ for the maximum R^2.[15]

For subperiods 2 to 9, the values for ϕ that give the maximum R^2 are 1.25, 4.75, 1.75, 47, 38, 0.25, 25, and -1.25, respectively, using R_{Ht} as the predictor. Using Q_{Ht}, the values of ϕ are 1.5, 4.25, 2, 20, 27, 0.75, 13, and 12, respectively. With these values of ϕ in hand, we have fully specified the hedging portfolio, its return and its dollar return. Table II reports the summary statistics for the return and dollar return on the hedging portfolio.

B. Optimal Forecasting Portfolios (OFPs)

Having constructed the return of the hedging portfolio in Subsection A, we wish to compare its forecast power to those of other forecasters. According to Proposition 2, the returns of the hedging portfolio should outperform the returns of any other portfolio in predicting future market returns. Specifically, if we regress R_{Mt} on the lagged return of any arbitrary portfolio p, the $\bar{R}^2$ should be no greater than that of (43).

It is impractical to compare (43) against all possible portfolios, and uninformative to compare it against random portfolios. Instead, we need only make comparisons against "optimal forecast portfolios," portfolios that are optimal

[14] This approach ignores the impact of statistical variation on the "optimal" ϕ, which is beyond the scope of this paper but is explored further in related contexts by Foster, Smith, and Whaley (1997) and Lo and MacKinlay (1997).

[15] There is no restriction on the value of ϕ if we use dollar returns. We do consider wider ranges of ϕ, but we do not find better candidates for the hedging portfolio in most cases. Even if we did, the identified hedging portfolio would involve large short positions in certain stocks, which may be deemed unrealistic. In any case, limiting ourselves to a smaller range of ϕ biases against our finding supporting evidence for the model. The empirical results presented in the paper can therefore be viewed as a "lower bound." See Lo and Wang (2005) for more details.

Table II

Summary statistics for the returns and dollar returns of the hedging portfolio constructed from individual stocks' volume data using weekly returns and volume data for NYSE and AMEX stocks from 1962 to 2004 and for subperiods. Autocorrelation of lag k is denoted by ρ_k.

	Sample Period								
Statistic	Entire	67–71	72–76	77–81	82–86	87–91	92–96	97–01	02–04
				Hedging Portfolio Return, R_{Ht}					
Mean (%)	1.2	0.1	0.5	0.7	1.1	5.2	0.3	0.6	0.5
S.D. (%)	18.0	2.9	3.9	4.5	4.6	47.7	1.3	8.0	9.4
Skewness	25.2	0.6	0.5	−0.3	0.3	10.2	−0.2	−0.2	0.0
Kurtosis	860.8	1.5	7.6	0.7	1.3	130.5	0.9	0.0	6.2
ρ_1 (%)	1.3	19.9	14.1	19.6	12.5	0.4	−16.5	−10.4	−4.2
ρ_2 (%)	−5.8	1.8	0.6	7.1	3.6	−7.0	−2.8	−3.3	−9.4
ρ_3 (%)	10.4	−2.8	−3.6	−1.0	7.3	9.9	−0.3	8.3	9.2
ρ_4 (%)	17.0	7.0	4.3	4.5	−11.3	18.2	−1.0	−4.5	−21.6
ρ_5 (%)	−7.9	11.4	14.4	−2.6	−10.3	−9.9	−2.5	−0.8	13.2
				Hedging Portfolio Dollar Return, Q_{Ht}					
Mean	1.8	0.1	1.2	2.3	5.6	3.2	0.3	0.7	0.0
S.D.	20.0	3.6	11.1	21.5	25.4	20.9	1.8	35.9	10.5
Skewness	0.0	0.2	−0.1	−0.5	−0.1	2.1	0.2	−0.3	0.5
Kurtosis	10.5	−0.1	0.5	2.3	6.5	13.3	2.0	2.0	1.0
ρ_1 (%)	2.8	21.9	25.1	20.0	9.8	15.7	−12.2	−15.2	−3.8
ρ_2 (%)	4.4	1.4	14.8	5.2	12.5	−1.5	−9.5	−0.7	−4.3
ρ_3 (%)	6.5	0.3	7.7	1.0	7.1	−4.1	3.7	9.9	0.8
ρ_4 (%)	−0.8	6.1	8.4	12.7	−3.7	−6.6	1.4	−5.1	4.6
ρ_5 (%)	1.2	11.6	10.2	−0.2	5.1	−1.6	−2.7	−2.0	−1.2

forecasters of R_{Mt}, since by construction, no other portfolios can have higher levels of predictability than these. The following proposition shows how to construct optimal forecasting portfolios (OFPs):

PROPOSITION 5: *Let Γ_0 and Γ_1 denote the contemporaneous and first-order autocovariance matrix of the vector of all returns. For any arbitrary target portfolio q with weights $w_q = (w_{q1}; \ldots ; w_{qN})$, define $A \equiv \Gamma_0^{-1}\Gamma_1 w_q w_q' \Gamma_1'$. The optimal forecast portfolio of w_q is given by the normalized eigenvector of A that corresponds to its largest eigenvalue.*[16]

Since Γ_0 and Γ_1 are unobservable, they must be estimated using historical data. Given the large number of stocks in our sample (over 2,000 in each subperiod) and the relatively short time series in each subperiod (261 weekly observations), the standard estimators for Γ_0 and Γ_1 become singular. However, it is possible to construct OFPs from a much smaller number of "basis portfolios," and then compare the predictive power of these OFPs to the hedging portfolio. As long as the basis portfolios are not too specialized, the $\bar{R}^2$s are likely to be similar to those obtained from the entire universe of all stocks.

[16] When returns are driven by the process given in (12), the OFP given here is exactly the hedging portfolio, as shown earlier. See the proof of this proposition in the Appendix.

Our general approach is to evaluate the forecast power of the hedging portfolio in 5-year subperiods (testing periods), using preceding 5-year subperiods (estimation periods) to estimate the OFPs.

We form several sets of basis portfolios by sorting all the J stocks into K groups of equal numbers ($K \leq J$) according to market capitalization, market beta, and SIC codes, and then construct value-weighted portfolios within each group.[17] This procedure yields K basis portfolios for which the corresponding Γ_0 and Γ_1 can be estimated using the portfolios' weekly returns within each subperiod. Based on the estimated autocovariance matrices, the OFP can be computed easily according to Proposition 5.

In selecting the number of basis portfolios K, we face the following trade-off: Fewer portfolios yields better sampling properties for the covariance matrix estimators, but less desirable properties for the OFP since the predictive power of the OFP is obviously maximized when $K = J$. As a compromise, for the OFPs based on market capitalization and market betas, we choose K to be 10, 15, 20, and 25. For the OFP based on SIC codes, we choose 13 industry groupings that we describe in more detail below.

Specifically, for each 5-year test period in which we wish to evaluate the forecast power of the hedging portfolio, we use the previous 5-year estimation subperiod to estimate the OFPs. For the OFP based on 10 market-capitalization-sorted portfolios, which we call "CAP10," we construct 10 value-weighted portfolios each week, one for each market-capitalization decile. Market-capitalization deciles are recomputed each week, and the time series of decile returns form the 10 basis portfolio returns of CAP10, with which we can estimate Γ_0 and Γ_1. To compute the OFP, we also require the weights ω_q of the target portfolio, in this case the market portfolio. Since the testing period follows the estimation period, we use the market capitalization of each group in the last week of the estimation period to map the weights of the market portfolio into a 10×1 vector of weights for the 10 basis portfolios. The weights of the OFP for the CAP10 basis portfolios follow immediately from Proposition 5. The same procedure is used to form OFPs for CAP15, CAP20, and CAP25 basis portfolios.

The OFPs of market-beta-sorted basis portfolios are constructed in a similar manner. We first estimate the market betas of individual stocks in the estimation period, sorting them according to their estimated betas, and then form small groups of basis portfolios, calculating value-weighted returns for each group. We consider 10, 15, 20, and 25 groups, denoted by "BETA10," "BETA15," and so on. The same procedure is then followed to construct the OFPs for each of these sets of basis portfolios.

Finally, the industry portfolios are based on SIC-code groupings. The first two digits of the SIC code yield 60 to 80 industry categories, depending on the

[17] It is important that we use value-weighted portfolios here so that the market portfolio, whose return we wish to predict, is a portfolio of these basic portfolios (recall that the target portfolio ω_q that we wish to forecast is a linear combination of the vector of returns for which Γ_k is the k^{th}-order autocovariance matrix).

sample period, and some of the categories contain only one or two stocks. On the other hand, the first digit yields only 8 broad industry categories. As a compromise, we use a slightly more disaggregated grouping of 13 industries, given by the following correspondence:[18]

no.	SIC Codes	Description
1	1–14	Agriculture, forest, fishing, mining
2	15–19, 30, 32–34	Construction, basic materials (steel, glass, concrete, etc.)
3	20–21	Food and tobacco
4	22, 23, 25, 31, 39	Textiles, clothing, consumer products
5	24, 26–27	Logging, paper, printing, publishing
6	28	Chemicals
7	29	Petroleum
8	35–36, 38	Machinery and equipment supply, including computers
9	37, 40–47	Transportation-related
10	48–49	Utilities and telecommunications
11	50–59	Wholesale distributors, retail
12	60–69	Financial
13	70–89, 98–99	Recreation, entertainment, services, conglomerates, etc.

Each week, stocks are sorted according to their SIC codes into the 13 categories defined above, and value-weighted returns are computed for each group, yielding the 13 basis portfolios that we denote by "SIC13." The autocovariance matrices are then estimated and the OFP constructed according to Proposition 5.

C. Hedging Portfolio Return as a Predictor of Market Returns

Table III reports the results of the regressions of R_{Mt} on various lagged OFP returns and on the lagged return and dollar return on the hedging portfolio, R_{Ht-1} and Q_{Ht-1}. For each subperiod, we use the hedging portfolio with the ϕ value determined in Subsection A. For completeness, we also include four additional regressions, with lagged value- and equal-weighted CRSP index returns, the logarithm of the reciprocal of lagged market capitalization, and the lagged 3-month constant-maturity Treasury bill return as predictors.[19] Table III shows

[18] We are grateful to Jonathan Lewellen for sharing his industry classification scheme.

[19] We also consider nine other interest-rate predictors (6-month and 1-year Treasury bill rates, 3-month, 6-month, and 1-year off-the-run Treasury bill rates, 1-month and 3-month CD and Eurodollar rates, and the Fed Funds rate, all obtained from the Federal Reserve Bank of St. Louis, http://www.stls.frb.org/fred/data/wkly.html. Each of these variables produces results similar to those for the 3-month constant-maturity Treasury bill return, hence we omit those regressions from Table III.

Table III

R^2s of linear forecasts of weekly market portfolio returns by lagged weekly returns of the optimal forecast portfolios (OFPs) for the set of 10, 15, 20, and 25 beta-sorted portfolios (BETA10, BETA15, BETA20, and BETA25, respectively), the OFPs for the set of 10, 15, 20, and 25 market-capitalization-sorted portfolios (CAP10, CAP15, CAP20, and CAP25, respectively), the SIC-sorted OFP (SIC13), the return and dollar return on the hedging portfolio (R_H and Q_H, respectively), minus log market capitalization ($\log(\text{Cap}^{-1})$), the lagged returns on the CRSP value- and equal-weighted portfolios (VW and EW, respectively) for subperiods from 1962 to 2004, and lagged constant-maturity (3-month) Treasury bill rates (TBill) for sub-periods from 1982 to 2004, in percentage. The values of ϕ for the return R_H and dollar return Q_H on the hedging portfolio are selected optimally for each subperiod.

	Sample Period							
Forecaster	67–71	72–76	77–81	82–86	87–91	92–96	97–01	02–04
BETA10	1.3	0.0	0.0	0.0	1.0	0.4	0.4	2.2
BETA15	0.1	0.5	0.3	0.3	1.5	0.0	0.4	1.0
BETA20	1.4	2.6	0.1	1.5	0.2	0.3	0.1	2.6
BETA25	1.2	0.1	0.1	0.5	0.3	0.3	0.4	1.6
CAP10	1.4	1.1	0.0	0.1	2.1	0.5	0.1	0.7
CAP15	0.5	0.2	0.0	0.0	0.0	0.0	0.3	0.3
CAP20	0.0	0.5	0.3	0.2	0.0	0.0	0.0	2.5
CAP25	0.5	0.1	0.4	0.1	0.0	0.1	1.2	0.9
SIC13	3.1	0.1	0.5	1.4	2.1	0.2	0.2	3.4
R_H	4.5	0.8	1.3	1.2	7.3	3.2	3.9	5.1
Q_H	5.6	1.4	1.2	0.9	2.4	2.2	4.1	1.7
$\log(\text{Cap}^{-1})$	2.1	0.8	0.2	0.5	0.8	0.3	1.9	0.9
VW	3.7	0.0	0.5	0.5	0.3	2.8	0.6	2.1
EW	1.6	0.3	0.7	0.3	0.1	4.1	0.7	4.6
TBill				0.6	0.3	1.1	0.0	0.4

that the hedging portfolios outperform all of the other competing portfolios in forecasting future market returns in 5 of the 8 subperiods (subperiods 2, 4, 6, 8, and 9). In subperiod 3, only one OFP (BETA20) outperforms the hedging portfolio, and in subperiod 5, BETA20 and SIC13's OFPs outperform the hedging portfolio, but only marginally. In subperiod 7, the equal-weighted CRSP index return outperforms the hedging portfolio.

However, several caveats should be kept in mind with regard to the three subperiods in which the hedging portfolios are surpassed by one or two competing portfolios. First, in these three subperiods, the hedging portfolio still outperforms most of the other competing portfolios. Second, there is no consistent winner in these subperiods. Third, the performance of the hedging portfolios is often close to the best performer. Moreover, the best performers in these subperiods performed poorly in the other subperiods, raising the possibility that their performance might be due to sampling variation. In contrast, the hedging portfolios forecasted the market return consistently in every subperiod. Indeed, among all of the regressors, the hedging portfolios are the most consistent across all 8 subperiods, a remarkable confirmation of the properties of the model of Sections I and II.[20]

V. The Hedging-Portfolio Return as a Risk Factor

To evaluate the success of the hedging portfolio return as a risk factor in the cross section of expected returns, we implement a slightly modified version of the well known regression tests outlined in Fama and MacBeth (1973). The basic approach is the same: Form portfolios sorted by an estimated parameter such as market beta coefficients in one time period (the "portfolio-formation period"), estimate betas for those same portfolios in a second nonoverlapping time period (the "estimation period"), and perform a cross-sectional regression test for the explanatory power of those betas using the returns of a third nonoverlapping time period (the "testing period"). However, in contrast to Fama and MacBeth (1973), we use weekly instead of monthly returns, and our portfolio formation, estimation, and testing periods are 5 years each.[21]

Specifically, we run the following bivariate regression for each security in the portfolio formation period, using only those securities that exist in all three periods:[22]

$$R_{jt} = \alpha_j + \beta_j^M R_{Mt} \; + \; \beta_j^H R_{Ht} + \varepsilon_{it}, \tag{44}$$

where R_{Mt} is the return on the CRSP value-weighted index and R_{Ht} is the return on the hedging portfolio. Using the estimated coefficients $\{\hat{\beta}_i^M\}$ and $\{\hat{\beta}_i^H\}$, we perform a double sort among the individual securities in the estimation period, creating 100 portfolios corresponding to the deciles of the estimated market and hedging portfolio betas. We re-estimate the two betas for each of these 100 portfolios in the estimation period, and use these estimated betas as regressors in the testing period, for which we estimate the following cross-sectional regression:

$$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_p^M + \gamma_{2t}\hat{\beta}_p^H + \eta_{pt}, \tag{45}$$

[20] On the other hand, the results in Table III must be tempered by the fact that the OFPs are only as good as the basis portfolios from which they are constructed. Increasing the number of basis portfolios should, in principle, increase the predictive power of the OFP. However, as the number of basis portfolios increases, the estimation errors in the autocovariance estimators $\hat{\gamma}_0$ and $\hat{\gamma}_1$ also increase for a fixed set of time-series observations, thus the impact on the predictive power of the OFP is not clear.

[21] Our first portfolio formation period, from 1962 to 1966, is only $4\frac{1}{2}$ years because the CRSP Daily Master file begins in July 1962. Fama and MacBeth's (1973) original procedure uses a 7-year portfolio formation period, a 5-year estimation period, and a 4-year testing period.

[22] This induces a certain degree of survivorship bias since our sample requires that stocks be listed for at least 15 years. While survivorship bias has a clear impact on expected returns and on the size effect, its implications for the cross-sectional explanatory power of the hedging portfolio are less obvious, hence we proceed cautiously with this selection criterion.

where R_{pt} is the equal-weighted portfolio return for securities in portfolio p, $p = 1, \ldots, 100$, which we construct from the double-sorted rankings of the portfolio estimation period, and $\hat{\beta}_{pt}^{M}$ and $\hat{\beta}_{pt}^{H}$ are the market and hedging-portfolio returns, respectively, of portfolio p that we obtain from the estimation period. This cross-sectional regression is estimated for each of the 261 weeks in the 5-year testing period, yielding a time series of coefficients $\{\hat{\gamma}_{0t}\}$, $\{\hat{\gamma}_{1t}\}$, and $\{\hat{\gamma}_{2t}\}$. This entire procedure is repeated by incrementing the portfolio formation, estimation, and testing periods by 5 years. We then perform the same analysis for the hedge portfolio dollar return series $\{Q_{Ht}\}$.

Because we use weekly instead of monthly data, it may be difficult to compare our results to other cross-sectional tests in the extant literature, for example, Fama and French (1992). Therefore, we apply our procedure to four other benchmark models: The standard CAPM in which R_{Mt} is the only regressor in (44); a two-factor model in which the hedging portfolio return factor is replaced by a "small-minus-big capitalization" or "SMB" portfolio return factor as in Fama and French (1992); a two-factor model in which the hedging-portfolio return factor is replaced by the OFP return factor described in Subsection B;[23] and a three-factor model in which both SMB and high-minus-low book-to-market (HML) factors are included along with the market factor, that is

$$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_{p}^{M} + \gamma_{2t}\hat{\beta}_{p}^{SMB} + \gamma_{3t}\hat{\beta}_{p}^{HML} + \eta_{pt}. \quad (46)$$

Table IV reports correlations in subperiods 1, 8, and 9 among the different portfolio return factors, returns on CRSP value- and equal-weighted portfolios, the return and dollar return on the hedging portfolio, returns on the SMB portfolio, the return on OFP BETA20, and the two turnover indices (see Lo and Wang (2005) for summary statistics for the return betas from the six linear factor models).

Table V summarizes the results of all of these cross-sectional regression tests for each of the seven testing periods from 1972 to 2004. In the first subpanel, which corresponds to the first testing period from 1972 to 1976, there is little evidence in support of the CAPM or any of the other linear models we estimate.[24] For example, the first three rows show that the time-series average of the market beta coefficients, $\{\hat{\gamma}_{1t}\}$, is 0.0%, with a t-statistic of 0.35 and an average $\bar{R}^2$ of 10.0%.[25] When the hedging portfolio beta $\hat{\beta}_{t}^{H}$ is added to the

[23] Specifically, the SMB portfolio return is constructed by taking the difference of the value-weighted returns of securities with market capitalization below and above the median market capitalization at the start of the 5-year subperiod.

[24] The two-factor model with OFP as the second factor is not estimated until the second testing period because we use the 1962 to 1966 period to estimate the covariances from which the OFP returns in the 1967 to 1971 period are constructed. Therefore, the OFP returns are not available in the first portfolio formation period.

[25] The t-statistic is computed under the assumption of independently and identically distributed coefficients $\{\gamma_{1t}\}$, which may not be appropriate. However, since this has become the standard method for reporting the results of these cross-sectional regression tests, we follow this convention to make our results comparable to those in the literature.

Table IV

Correlation matrix for weekly returns on the CRSP value-weighted index (R_{VWt}), the CRSP equal-weighted index (R_{EWt}), the hedging portfolio return (R_{Ht}), the hedging portfolio dollar return (Q_{Ht}), the return of the small-minus-big capitalization stocks portfolio (R_{SMBt}), the return of the high-minus-low book-to-market stocks portfolio (R_{HMLt}), the return R_{OFPt} of the optimal forecast portfolio (OFP) for the set of 20 market-beta-sorted basis portfolios, and the equal-weighted and share-weighted turnover indices (τ_t^{EW} and τ_t^{SW}), using CRSP weekly returns and volume data for NYSE and AMEX stocks for three subperiods: January 1967 – December 1971, January 1997 – December 2001, and January 2002 – December 2004.

	R_{VWt}	R_{EWt}	R_{Ht}	Q_{Ht}	R_{SMBt}	R_{HMLt}	R_{OFPt}	τ_t^{EW}	τ_t^{SW}
				January 1967 – December 1971 (261 Weeks)					
R_{VWt}	100.0	92.6	95.6	91.5	62.7	−44.1	−76.2	19.1	26.3
R_{EWt}	92.6	100.0	92.3	88.4	84.5	−40.8	−71.9	32.8	36.9
R_{Ht}	95.6	92.3	100.0	97.4	70.7	−52.4	−65.0	22.0	29.6
Q_{Ht}	91.5	88.4	97.4	100.0	69.8	−49.4	−60.1	22.9	29.8
R_{SMBt}	62.7	84.5	70.7	69.8	100.0	−40.6	−46.6	39.7	38.2
R_{HMLt}	−44.1	−40.8	−52.4	−49.4	−40.6	100.0	14.5	−9.0	−15.0
R_{OFPt}	−76.2	−71.9	−65.0	−60.1	−46.6	14.5	100.0	−7.5	−10.4
τ_t^{EW}	19.1	32.8	22.0	22.9	39.7	−9.0	−7.5	100.0	93.1
τ_t^{SW}	26.3	36.9	29.6	29.8	38.2	−15.0	−10.4	93.1	100.0
				January 1997 – December 2001 (261 Weeks)					
R_{VWt}	100.0	80.4	57.9	43.4	−20.7	−48.7	−45.7	3.3	−0.9
R_{EWt}	80.4	100.0	54.8	42.6	24.9	−46.7	−33.1	7.9	−1.4
R_{Ht}	57.9	54.8	100.0	89.2	17.0	−71.0	−15.5	0.0	−2.2
Q_{Ht}	43.4	42.6	89.2	100.0	25.3	−70.0	−1.2	3.8	0.4
R_{SMBt}	−20.7	24.9	17.0	25.3	100.0	−41.8	25.8	7.0	0.1
R_{HMLt}	−48.7	−46.7	−71.0	−70.0	−41.8	100.0	2.5	−0.1	4.4
R_{OFPt}	−45.7	−33.1	−15.5	−1.2	25.8	2.5	100.0	−0.9	1.8
τ_t^{EW}	3.3	7.9	0.0	3.8	7.0	−0.1	−0.9	100.0	92.4
τ_t^{SW}	−0.9	−1.4	−2.2	0.4	0.1	4.4	1.8	92.4	100.0
				January 2002 – December 2004 (156 Weeks)					
R_{VWt}	100.0	91.8	68.6	−11.5	−17.1	−6.6	55.0	9.3	4.1
R_{EWt}	91.8	100.0	76.3	−32.7	15.8	10.2	50.3	5.5	−3.1
R_{Ht}	68.6	76.3	100.0	−60.4	14.1	−2.0	31.5	5.4	−0.4
Q_{Ht}	−11.5	−32.7	−60.4	100.0	−56.5	−2.4	−2.5	−3.9	3.3
R_{SMBt}	−17.1	15.8	14.1	−56.5	100.0	9.4	−7.2	−5.6	−9.1
R_{HMLt}	−6.6	10.2	−2.0	−2.4	9.4	100.0	−17.2	−18.3	−27.9
R_{OFPt}	55.0	50.3	31.5	−2.5	−7.2	−17.2	100.0	12.1	3.7
τ_t^{EW}	9.3	5.5	5.4	−3.9	−5.6	−18.3	12.1	100.0	77.8
τ_t^{SW}	4.1	−3.1	−0.4	3.3	−9.1	−27.9	3.7	77.8	100.0

regression, the $\bar{R}^2$ increases to 14.3% but the average of the coefficients $\{\hat{\gamma}_{2t}\}$ is −0.2% with a t-statistic of −0.82. The average market beta coefficient is still insignificant, but it has now switched sign. The results for the two-factor model with the hedging portfolio dollar return factor and the two-factor model with the SMB factor are similar. The three-factor model is even less successful, with statistically insignificant coefficients close to 0.0% and an average $\bar{R}^2$ of 8.8%.

In the second testing period, both specifications with the hedging portfolio factor exhibit statistically significant means for the hedging portfolio betas,

Table V

Cross-sectional regression tests of six linear factor models along the lines of Fama and MacBeth (1973), using weekly returns for NYSE and AMEX stocks from 1962 to 2004 in 5-year subperiods for the portfolio formation, estimation, and testing periods, and 100 portfolios in the cross-sectional regressions each week. The six linear factor models are: the standard CAPM ($\hat{\beta}_p^M$); four two-factor models in which the first factor is the market beta and the second factors are, respectively, the hedging portfolio return beta ($\hat{\beta}_p^{HR}$), the hedging portfolio dollar return beta ($\hat{\beta}_p^{HQ}$), the beta of a small-minus-big cap portfolio return ($\hat{\beta}_p^{SMB}$), the beta of the Fama and French (1992) high-minus-low book-to-market portfolio return ($\hat{\beta}_p^{HML}$), and the beta of the optimal forecast portfolio based on a set of 25 market-beta-sorted basis portfolios ($\hat{\beta}_p^{OFP}$); and a three-factor model with $\hat{\beta}_p^M$, $\hat{\beta}_p^{SMB}$, and $\hat{\beta}_p^{HML}$ as the three factors.

Model	Statistic	$\hat{\gamma}_{0t}$	$\hat{\gamma}_{1t}$	$\hat{\gamma}_{2t}$	$\hat{\gamma}_{3t}$	$\overline{R}^2(\%)$
January 1972 – December 1976 (261 Weeks)						
$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_p^M + \varepsilon_{pt}$	Mean (%):	0.2	0.0			10.0
	S.D. (%):	1.5	2.1			10.9
	t-stat:	1.64	0.35			
$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_p^M + \gamma_{2t}\hat{\beta}_p^{HR} + \varepsilon_{pt}$	Mean (%):	0.4	−0.2	−0.2		14.3
($\phi = 1.25$)	S.D. (%):	3.5	3.5	3.7		10.9
	t-stat:	2.04	−1.05	−0.82		
$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_p^M + \gamma_{2t}\hat{\beta}_p^{HQ} + \varepsilon_{pt}$	Mean (%):	0.4	−0.2	−10.4		15.5
($\phi = 1.50$)	S.D. (%):	3.2	3.4	379.7		10.9
	t-stat:	2.16	−1.08	−0.44		
$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_p^M + \gamma_{2t}\hat{\beta}_p^{SMB} + \varepsilon_{pt}$	Mean (%):	0.1	0.0	6.3		12.1
	S.D. (%):	1.4	2.4	114.2		10.8
	t-stat:	1.42	0.22	0.90		
$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_p^M + \gamma_{2t}\hat{\beta}_p^{HML} + \varepsilon_{pt}$	Mean (%):	0.0	0.1	0.2		7.8
	S.D. (%):	1.8	1.9	1.5		7.8
	t-stat:	0.40	1.08	1.66		
$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_p^M + \gamma_{2t}\hat{\beta}_p^{SMB} + \gamma_{3t}\hat{\beta}_p^{HML} + \varepsilon_{pt}$	Mean (%):	0.0	0.1	0.0	0.1	8.8
	S.D. (%):	1.8	2.1	1.2	1.5	7.6
	t-stat:	0.28	0.80	0.50	1.27	
January 1977 – December 1981 (261 Weeks)						
$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_p^M + \varepsilon_{pt}$	Mean (%):	0.1	0.3			11.7
	S.D. (%):	1.1	2.2			12.8
	t-stat:	1.17	2.57			
$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_p^M + \gamma_{2t}\hat{\beta}_p^{HR} + \varepsilon_{pt}$	Mean (%):	0.3	−0.1	−1.2		13.1
($\phi = 4.75$)	S.D. (%):	1.4	2.0	5.1		12.4
	t-stat:	3.75	−0.90	−3.71		
$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_p^M + \gamma_{2t}\hat{\beta}_p^{HQ} + \varepsilon_{pt}$	Mean (%):	0.3	−0.1	−156.4		12.5
($\phi = 4.25$)	S.D. (%):	1.3	2.0	610.4		12.2
	t-stat:	3.91	−0.75	−4.14		
$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_p^M + \gamma_{2t}\hat{\beta}_p^{SMB} + \varepsilon_{pt}$	Mean (%):	0.1	0.0	29.9		14.9
	S.D. (%):	1.1	1.7	108.8		13.4
	t-stat:	2.25	−0.16	4.43		
$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_p^M + \gamma_{2t}\hat{\beta}_p^{HML} + \varepsilon_{pt}$	Mean (%):	0.2	0.2	0.1		9.3
	S.D. (%):	1.3	1.9	0.9		9.2
	t-stat:	2.15	1.52	1.58		
$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_p^M + \gamma_{2t}\hat{\beta}_p^{OFP} + \varepsilon_{pt}$	Mean (%):	0.3	0.1	0.1		14.1
	S.D. (%):	1.8	2.3	3.6		11.6
	t-stat:	2.74	0.84	0.63		
$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_p^M + \gamma_{2t}\hat{\beta}_p^{SMB} + \gamma_{3t}\hat{\beta}_p^{HML} + \varepsilon_{pt}$	Mean (%):	0.2	−0.0	0.3	−0.1	11.5
	S.D. (%):	1.2	1.7	1.0	1.0	10.0
	t-stat:	2.72	−0.07	4.61	−0.85	

(continued)

Table V—*Continued*

January 1982 – December 1986 (261 Weeks)						
$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_p^M + \varepsilon_{pt}$	Mean (%):	0.6	−0.1			9.4
	S.D. (%):	1.1	1.9			11.1
	t-stat:	8.17	−1.04			
$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_p^M + \gamma_{2t}\hat{\beta}_p^{HR} + \varepsilon_{pt}$	Mean (%):	0.6	−0.1	−0.6		9.6
($\phi = 1.75$)	S.D. (%):	1.1	2.0	5.5		9.4
	t-stat:	8.39	−0.78	−1.73		
$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_p^M + \gamma_{2t}\hat{\beta}_p^{HQ} + \varepsilon_{pt}$	Mean (%):	0.6	−0.2	−74.0		10.4
($\phi = 2.00$)	S.D. (%):	1.1	1.9	1987.4		9.5
	t-stat:	8.36	−1.30	−0.60		
$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_p^M + \gamma_{2t}\hat{\beta}_p^{SMB} + \varepsilon_{pt}$	Mean (%):	0.5	−0.2	3.8		10.0
	S.D. (%):	1.2	1.9	115.4		8.4
	t-stat:	7.45	−1.26	0.53		
$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_p^M + \gamma_{2t}\hat{\beta}_p^{HML} + \varepsilon_{pt}$	Mean (%):	0.5	−0.1	0.2		9.2
	S.D. (%):	1.4	1.9	1.5		8.5
	t-stat:	5.57	−0.83	1.81		
$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_p^M + \gamma_{2t}\hat{\beta}_p^{OFP} + \varepsilon_{pt}$	Mean (%):	0.5	−0.1	0.0		11.7
	S.D. (%):	1.1	2.0	2.1		10.8
	t-stat:	7.55	−0.82	0.20		
$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_p^M + \gamma_{2t}\hat{\beta}_p^{SMB} + \gamma_{3t}\hat{\beta}_p^{HML} + \varepsilon_{pt}$	Mean (%):	0.4	−0.0	0.0	0.1	10.0
	S.D. (%):	1.4	2.2	1.0	1.1	8.5
	t-stat:	4.49	−0.24	0.55	1.97	
January 1987 – December 1991 (261 Weeks)						
$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_p^M + \varepsilon_{pt}$	Mean (%):	0.2				5.9
	S.D. (%):	1.3	2.3			8.7
	t-stat:	2.65	0.20			
$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_p^M + \gamma_{2t}\hat{\beta}_p^{HR} + \varepsilon_{pt}$	Mean (%):	0.2	0.0	0.0		5.4
($\phi = 47$)	S.D. (%):	1.6	1.9	6.0		6.1
	t-stat:	2.25	0.11	0.13		
$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_p^M + \gamma_{2t}\hat{\beta}_p^{HQ} + \varepsilon_{pt}$	Mean (%):	0.2	0.0	18.9		6.0
($\phi = 20$)	S.D. (%):	1.6	1.9	1819.4		6.7
	t-stat:	2.43	−0.15	0.17		
$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_p^M + \gamma_{2t}\hat{\beta}_p^{SMB} + \varepsilon_{pt}$	Mean (%):	0.3	0.0	−7.5		7.8
	S.D. (%):	1.4	2.0	123.5		8.2
	t-stat:	3.10	0.16	−0.98		
$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_p^M + \gamma_{2t}\hat{\beta}_p^{HML} + \varepsilon_{pt}$	Mean (%):	0.2	0.0	0.0		6.3
	S.D. (%):	1.6	2.0	1.8		7.6
	t-stat:	2.11	0.03	−0.38		
$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_p^M + \gamma_{2t}\hat{\beta}_p^{OFP} + \varepsilon_{pt}$	Mean (%):	0.3	−0.1	0.0		6.4
	S.D. (%):	1.5	2.1	2.1		7.3
	t-stat:	2.73	−0.39	−0.23		
$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_p^M + \gamma_{2t}\hat{\beta}_p^{SMB} + \gamma_{3t}\hat{\beta}_p^{HML} + \varepsilon_{pt}$	Mean (%):	0.3	0.0	−0.1	−0.1	7.8
	S.D. (%):	1.9	2.1	1.2	1.5	6.8
	t-stat:	2.21	0.16	−0.97	−1.10	

(continued)

with average coefficients and t-statistics of −1.2% and −3.71 for the hedging-portfolio return factor and −1.56 and −4.14 for the hedging portfolio dollar return factor, respectively. In contrast, the market beta coefficients are not significant in either of these specifications, and are also of the wrong sign. The only other specifications with a significant mean coefficient are the two-factor model with SMB as the second factor (with an average coefficient of 29.9% for

Table V—*Continued*

January 1992 – December 1996 (261 Weeks)						
$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_p^M + \varepsilon_{pt}$	Mean (%):	0.2	0.1			5.7
	S.D. (%):	1.3	2.0			7.7
	t-stat:	2.68	1.18			
$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_p^M + \gamma_{2t}\hat{\beta}_p^{HR} + \varepsilon_{pt}$	Mean (%):	0.2	0.1	−0.4		6.9
($\phi = 38$)	S.D. (%):	1.3	2.0	9.1		6.8
	t-stat:	2.79	1.16	−0.65		
$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_p^M + \gamma_{2t}\hat{\beta}_p^{HQ} + \varepsilon_{pt}$	Mean (%):	0.3	0.0	−158.4		6.2
($\phi = 27$)	S.D. (%):	1.5	2.2	1299.2		6.6
	t-stat:	3.28	−0.18	−1.97		
$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_p^M + \gamma_{2t}\hat{\beta}_p^{SMB} + \varepsilon_{pt}$	Mean (%):	0.2	0.1	15.4		6.7
	S.D. (%):	1.5	1.9	115.7		7.0
	t-stat:	1.65	0.86	2.15		
$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_p^M + \gamma_{2t}\hat{\beta}_p^{HML} + \varepsilon_{pt}$	Mean (%):	0.3	0.1	0.0		7.7
	S.D. (%):	1.4	1.9	1.0		7.9
	t-stat:	3.19	0.78	−0.43		
$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_p^M + \gamma_{2t}\hat{\beta}_p^{OFP} + \varepsilon_{pt}$	Mean (%):	0.1	0.2	0.2		7.9
	S.D. (%):	1.6	2.0	1.5		7.4
	t-stat:	0.90	1.24	2.41		
$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_p^M + \gamma_{2t}\hat{\beta}_p^{SMB} + \gamma_{3t}\hat{\beta}_p^{HML} + \varepsilon_{pt}$	Mean (%):	0.3	0.0	0.2	−0.0	6.6
	S.D. (%):	1.5	2.0	1.1	1.1	5.8
	t-stat:	2.76	0.08	2.21	−0.27	
January 1997 – December 2001 (261 Weeks)						
$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_p^M + \varepsilon_{pt}$	Mean (%):	0.3	0.0			8.1
	S.D. (%):	1.4	2.5			9.9
	t-stat:	2.98	0.27			
$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_p^M + \gamma_{2t}\hat{\beta}_p^{HR} + \varepsilon_{pt}$	Mean (%):	0.3	0.0	0.0		10.3
($\phi = 1.75$)	S.D. (%):	1.4	2.2	2.0		10.4
	t-stat:	3.35	−0.05	−0.21		
$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_p^M + \gamma_{2t}\hat{\beta}_p^{HQ} + \varepsilon_{pt}$	Mean (%):	0.3	0.0	−17.9		9.4
($\phi = 2.00$)	S.D. (%):	1.7	2.3	307.4		8.5
	t-stat:	2.86	−0.23	−0.94		
$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_p^M + \gamma_{2t}\hat{\beta}_p^{SMB} + \varepsilon_{pt}$	Mean (%):	0.3	−0.1	0.1		9.3
	S.D. (%):	1.6	2.4	1.3		8.3
	t-stat:	3.58	−0.97	1.76		
$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_p^M + \gamma_{2t}\hat{\beta}_p^{HML} + \varepsilon_{pt}$	Mean (%):	0.3	0.0	−0.1		8.8
	S.D. (%):	1.5	2.5	1.8		8.4
	t-stat:	3.35	−0.05	−0.78		
$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_p^M + \gamma_{2t}\hat{\beta}_p^{OFP} + \varepsilon_{pt}$	Mean (%):	0.3	0.0	0.1		6.4
	S.D. (%):	1.5	2.2	1.6		7.5
	t-stat:	2.91	0.26	0.71		
$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_p^M + \gamma_{2t}\hat{\beta}_p^{SMB} + \gamma_{3t}\hat{\beta}_p^{HML} + \varepsilon_{pt}$	Mean (%):	0.4	−0.2	0.2	−0.0	9.3
	S.D. (%):	1.7	2.8	1.5	1.8	7.3
	t-stat:	4.31	−1.39	1.8	−0.37	

(continued)

the SMB factor and a *t*-statistic of 4.43) and the three-factor model in which the SMB factor is also significant (an average coefficient of 0.3% and a *t*-statistic of 4.61).

For the five remaining test periods, the only specifications with any statistically significant factors are the SMB and OFP two-factor models in the

Table V—*Continued*

January 2002 – December 2004 (156 Weeks)						
$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_p^M + \varepsilon_{pt}$	Mean (%):	0.4	0.0			3.9
	S.D. (%):	1.7	1.3			4.2
	t-stat:	3.26	−0.27			
$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_p^M + \gamma_{2t}\hat{\beta}_p^{HR} + \varepsilon_{pt}$	Mean (%):	0.6	−0.2	0.6		9.3
($\phi = 1.75$)	S.D. (%):	1.7	2.5	8.0		9.6
	t-stat:	4.54	−0.94	0.94		
$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_p^M + \gamma_{2t}\hat{\beta}_p^{HQ} + \varepsilon_{pt}$	Mean (%):	0.7	−0.3	−106.5		5.8
($\phi = 2.00$)	S.D. (%):	2.2	2.2	699.7		6.2
	t-stat:	4.29	−1.71	−1.90		
$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_p^M + \gamma_{2t}\hat{\beta}_p^{SMB} + \varepsilon_{pt}$	Mean (%):	0.5	−0.1	0.1		4.2
	S.D. (%):	2.2	0.7	0.6		4.9
	t-stat:	2.71	−1.43	1.59		
$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_p^M + \gamma_{2t}\hat{\beta}_p^{HML} + \varepsilon_{pt}$	Mean (%):	0.5	−0.1	0.0		6.1
	S.D. (%):	2.0	1.4	1.0		5.4
	t-stat:	3.17	−0.86	−0.55		
$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_p^M + \gamma_{2t}\hat{\beta}_p^{OFP} + \varepsilon_{pt}$	Mean (%):	0.4	0.0	0.0		4.1
	S.D. (%):	1.7	1.3	0.4		4.0
	t-stat:	3.15	0.01	−0.19		
$R_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_p^M + \gamma_{2t}\hat{\beta}_p^{SMB} + \gamma_{3t}\hat{\beta}_p^{HML} + \varepsilon_{pt}$	Mean (%):	0.4	0.0	0.0	0.1	5.8
	S.D. (%):	2.4	0.5	0.4	1.1	5.3
	t-stat:	2.03	0.03	1.57	0.75	

1992–1996 testing period though the hedging portfolio dollar return factor is close to significant in the 1997–2001 subperiod. However, the $\bar{R}^2$s in the last four testing periods are substantially lower than in the earlier periods, perhaps reflecting the greater volatility of equity returns in recent years.

Overall, the results do not provide overwhelming support for any factor in explaining the cross-sectional variation of expected returns. There is, of course, the ubiquitous problem of lack of power in these cross-sectional regression tests, hence we should not be surprised that no single factor stands out (see e.g., MacKinlay (1987, 1994)). However, the point estimates of the cross-sectional regressions show that the hedging portfolio factor is comparable both in magnitude and in performance to other commonly proposed factors, and yields a promising new direction for intertemporal asset pricing research.

VI. Conclusion

By deriving an explicit link between economic fundamentals and the dynamic properties of asset returns and volume, we have shown that interactions between prices and quantities in equilibrium yield a rich set of implications for any asset-pricing model. Indeed, by exploiting the relation between prices and volume in our dynamic equilibrium model, we are able to identify and construct the hedging portfolio that all investors use to hedge against changes in market conditions. Moreover, our empirical analysis shows that this hedging portfolio has considerable forecast power in predicting future returns of the market portfolio—a property of the true hedging portfolio—and its abilities to explain cross-sectional variation in expected returns is comparable to other popular risk factors such as market betas, the Fama and French (1992) SMB and HML factors, and optimal forecast portfolios.

Although, our model is purposefully parsimonious so as to focus attention on the essential features of risk-sharing and trading activity, it underscores the general point that quantities, together with prices, should be an integral part of any analysis of asset markets, both theoretically and empirically, as our results clearly indicate.

Appendix: Proofs

Proof of Theorem 1: We prove Theorem 1 by first deriving investor asset demand under the price function (9) and then solving the coefficient vectors a and b to clear the stock market.

For simplicity in notation, let $u_t \equiv (D_t; X_t)$, where $(\cdot\,;\cdot)$ denotes a column vector and $(\cdot\,,\cdot)$ denotes a row vector. From (9), we have

$$Q_{t+1} = \bar{Q}_t + \tilde{Q}_{t+1}, \tag{A1}$$

where $\bar{Q}_t = ra + (1{+}r)bX_t$, $\tilde{Q}_{t+1} = (1, -b)u_{t+1}$, and 1 is an $(j \times j)$ identity matrix.

We now consider investor i's optimal portfolio choice. Let S_t be the vector of his stock holding in period t. His next period wealth is $W_{t+1} = W_t + S_t{}'\bar{Q}_{t+1} + S_t{}'(1, -b)u_{t+1}$, where we have omitted superscript i for brevity. We also let $\lambda_{1t} \equiv \lambda_X X_t + \lambda_Y Y_t$ and $\lambda_{2t} \equiv \lambda_Z(1 + Z_t)$. Then,

$$\mathrm{E_t}\left[e^{-W_{t+1}-\lambda_{1t}D_{Mt+1}-\lambda_{2t}X_{t+1}}\right] = \mathrm{E_t}\left[e^{-W_t - S_t{}'\bar{Q}_t + (S_t+\lambda_{1t}\iota;-b'S_t+\lambda_{2t})'u_{t+1}}\right] \tag{A2}$$

$$= e^{-W_t - S_t{}'\bar{Q}_t + \frac{1}{2}\left(S_t + \lambda_{1t}\iota; -b'S_t + \lambda_{2t}\right)'\sigma\left(S_t + \lambda_{1t}\iota; -b'S_t + \lambda_{2t}\right)}, \tag{A3}$$

where σ is the covariance matrix of u_t. The investor's optimization problem then reduces to

$$\max_{S_t} \quad S_t'\bar{Q}_t - \frac{1}{2}\left(S_t + \lambda_{1t}\iota; -b'S_t + \lambda_{2t}\right)'\sigma\left(S_t + \lambda_{1t}\iota; -b'S_t + \lambda_{2t}\right). \tag{A4}$$

The first-order condition is

$$\begin{aligned} 0 = \bar{Q}_t &- \left(\sigma_{DD} - b\,\sigma_{DX}{}' - \sigma_{DX}b' + \sigma_{XX}b\,b'\right)S_t \\ &- \lambda_{1t}(\sigma_{DD} - b\,\sigma_{XD})\iota - \lambda_{2t}(\sigma_{DX} - b\,\sigma_{XX}). \end{aligned} \tag{A5}$$

The solution gives the investor's stock demand

$$S_t = \left(\sigma_{DD} - b\,\sigma_{DX}{}' - \sigma_{DX}b' + \sigma_{XX}b\,b'\right)^{-1}\left[\bar{Q}_t - \lambda_{1t}(\sigma_{DD} - b\,\sigma_{DZ})\iota - \lambda_{2t}(\sigma_{XD} - b\,\sigma_{XX})\right]. \tag{A6}$$

Summing (A5) over all investors and imposing the market-clearing condition, $\sum_i S_t^i = S_M$, we have

$$0 = [ra + (1+r)bX_t] - (1/I)\sigma_{QQ}S_M - \lambda^X \sigma_{QD}\iota X_t - \lambda_Z \sigma_{QX}. \qquad \text{(A7)}$$

It follows that

$$ra = (1/I)\sigma_{QQ}\, S_M + \lambda_Z\, \sigma_{QX} \qquad \text{(A8)}$$

$$(1+r)b = \lambda_X\, \sigma_{QD}\, S_M, \qquad \text{(A9)}$$

which uniquely determine the equilibrium a and b. Substituting (A7) into (A6), we obtain investor i's equilibrium stock holding

$$S_t^i = \left(I^{-1} - \lambda_Y Y_t^i\right)S_M - \left[\lambda_Y(b'S_M)Y_t^i + \lambda_Z Z_t^i\right](\sigma_{QQ})^{-1}\sigma_{QX}, \qquad \text{(A10)}$$

which is (10). Q.E.D.

Proof of Proposition 1: Let $x = \Delta h_{Mt}, y = \theta_H \Delta h_{Ht}$ (we omit superscripts i and j for notational simplicity), $x' = x/\sigma_x, y' = y/\sigma_y$, and $\eta = \sigma_y/\sigma_x$. We can write $|x+y| = \sigma_x|x' + \eta y'|$. Without loss of generality, we let $\sigma_x = 1$. Since Y_t and Z_t are normal, x' and y' are also jointly normal with unit standard deviations. We can write

$$|x + y| = |x' + \eta y'| = |x'| + \eta\, \mathrm{Sign}(x')y' + 1_{D(x',\eta y')}\delta(x', \eta y'), \qquad \text{(A11)}$$

where the last term is defined by the functions $D(z_1, z_2) = \{z_1 z_2 < 0, > |z_1| < |z_2|\}$ and $\delta(z_1, z_2) = -2|z_1| + 2|z_2|$. To assess the statistical error of the approximation

$$|x' + \eta y| \approx |x| + \eta\, \mathrm{Sign}(x)y, \qquad \text{(A12)}$$

we compute the expected approximation error using any loss function $L(|.|)$ that is nonnegative, nondecreasing on R^+, and of order $\alpha > 0$, that is, $\lim_{\epsilon\to 0} L(|\epsilon|)/\epsilon^\alpha = A$ exists and is finite. Loss functions satisfying these conditions include the mean absolute error ($L(z) = |z|, \alpha = 1$) and mean squared error ($L(z) = z^2, \alpha = 2$) loss functions. We then have

$$E[L(|1_{D(x,\eta y)}\delta(x, \eta y)|)] = P(D)E[L(|\delta(x, \eta y)|)|D], \qquad \text{(A13)}$$

where $P(D) = \int_0^\infty \int_0^{\eta y'} [f(-x', y') + f(x', -y')]\, dx'\, dy'$. Let ρ denote the correlation between $x\prime$ and y'. On D, we have $f(-x', y') \leq \frac{1}{2\pi} e^{-(1-\rho^2)y'^2/2}$. It immediately follows that $P(D)/\eta \leq c$, where c is a positive constant. We then have

$$E[L(|1_{D(x',\eta y')}\delta(x', \eta y')|)] < A\eta^{\alpha+1} + o(\eta^{\alpha+1}), \qquad \text{(A14)}$$

where A is a constant. Thus, even though the point-wise properties of the approximation are not satisfactory, the approximation error is small in a statistical sense for small η, that is, the expected loss decreases faster than $\eta^{\alpha+1}$. The definition of η shows that it depends the ratio of volatility of Δh_{Ht} to Δh_{Mt}. A sufficient condition is λ_X, $\lambda_Z/\lambda_Y \ll 1$. Q.E.D.

Proof of Proposition 2: Suppose we use the (dollar) return of portfolio S to predict future market returns. The resulting R^2 is

$$R^2 = \left(\mathrm{Cov}[(S'Q_t)Q_{Mt+1}]\right)^2 / \left(\mathrm{Var}[S'Q_t]\mathrm{Var}[Q_{Mt+1}]\right). \tag{A15}$$

To choose the S to maximize R^2, we solve the following problem

$$\max_S \quad S'\sigma_{QX}(b'S_M) \tag{A16}$$

$$\text{s.t.} \quad S'\sigma_{QQ}S = v. \tag{A17}$$

Up to a scaling constant, the solution is $S_H = (\sigma_{QQ})^{-1}\sigma_{QX}$. Q.E.D.

Proof of Proposition 3: When $X_t = 0, \forall t$, $Q_t = ra + \tilde{Q}_t = (S_M, -b)u_t$. Then, $\mathrm{Cov}[\tilde{Q}_t, \tilde{Q}_{Mt}] = \mathrm{Cov}[\tilde{Q}_t, S'_M\tilde{Q}_t] = \sigma_{DD}S_M$, $\mathrm{Var}[\tilde{Q}_{Mt}] = S'_M\sigma_{DD}S_M$, and (22) follows. Since $\sigma_{QX} = 0$ in this case, $\sigma_{MH} = 0$ and $\bar{Q}_M = (1/I)\sigma_M^2$. Thus, $\bar{Q} = \beta_M\bar{Q}_M$ which is (24). Q.E.D.

Proof of Proposition 4: Equation (25) simply follows from the joint normality of $\tilde{Q}_{t+1}$, $\tilde{Q}_{Mt+1}$, and $\tilde{Q}_{Ht+1}$. Equation (28) can be verified by substituting in the expressions for β_M, β_M, $\bar{Q}_M$, and $\bar{Q}_H$, which gives (21c). Q.E.D.

Proof of Proposition 5: Let Q_t denote the (dollar) return vector of a set of assets, $\Gamma_0 \equiv \mathrm{Cov}[Q_tQ_t{}']$ and $\Gamma_1 \equiv \mathrm{Cov}[Q_{t+1}Q_t{}']$. Suppose that S_a is a target portfolio and S_b is a forecasting portfolio whose return is used as a forecast for the return on S_a next period. Then, the R^2 from the forecast is

$$R^2 = \frac{(\mathrm{Cov}[S'_bQ_t, S'_aQ_{t+1}])^2}{\mathrm{Var}[S'_bQ_t]\mathrm{Var}[S'_aQ_{t+1}]} = \frac{(S'_b\Gamma_1S_a)^2}{(S'_a\Gamma_0S_a)(S'_b\Gamma_1S_b)}. \tag{A18}$$

The OFP for a given S_a is the portfolio that gives the highest R^2. In other words,

$$S_{\mathrm{OFP}} = \arg\max R^2 = \arg\max \frac{(S'_b\Gamma_1S_a)^2}{(S'_a\Gamma_0S_a)(S'_b\Gamma_0S_b)}. \tag{A19}$$

The optimality condition becomes

$$\left(\Gamma_0^{-1}\Gamma_1S_aS'_a\Gamma'_1\right) S_{\mathrm{OFP}} = R^2(S'_a\Gamma_0S_a)\, S_{\mathrm{OFP}}. \tag{A20}$$

It then immediately follows that S_{OFP} is the eigenvector corresponding to the largest eigenvalue of matrix $\Gamma_0^{-1}\Gamma_1S_aS_a{}'\Gamma'_1$, which should equal $R^2(S_a{}'\Gamma_0S_a)$.

Although, we derive the result in Proposition 5 using dollar returns and asset shares in defining portfolios, the same result holds if we use rates of return and dollar weights in defining portfolios, which is what the proposition states. Q.E.D.

The set of assets considered above does not have to be the entire universe of assets. In the event it is and expected asset returns are driven by a single state variable, as in (12), we have

$$\Gamma_1 = (1+r)\,\sigma_X\, b\, b' = \sigma_{QX}\sigma_{QX}'. \tag{A21}$$

It then follows that $S_{\text{OFP}} = S_H$ up to a scaling coefficient, as Proposition 2 shows.

REFERENCES

Abel, Andrew B., 1990, Asset prices under habit formation and catching up with the Joneses, *American Economic Review* 80, 38–42.

Andersen, Torben G., 1996, Return volatility and trading volume: An information flow interpretation, *Journal of Finance* 51, 169–204.

Archarya, Viral V., and Lasse H. Pedersen, 2002, Asset pricing with liquidity risk, *Journal of Financial Economics* 2005, 375–410.

Brennan, Michael, and Avanidhar Subrahmanyam, 1996, Market microstructure and asset pricing: On the compensation for illiquidity in stock returns, *Journal of Financial Economics* 41, 441–464.

Campbell, John, and John Cochrane, 1999, By force of habit: A consumption-based explanation of aggregate stock market behavior, *Journal of Political Economy* 107, 205–251.

Campbell, J., Grossman, S., and J. Wang, 1993. Trading volume and serial correlation in stock returns. *Quarterly Journal of Economics* 108, 905–939.

Chordia, Tarun, Richard Roll, and Avanidhar Subrahmanyam, 2000, Commonality in liquidity, *Journal of Financial Economics* 56, 3–28.

Fama, Eugene F., and Kenneth French, 1992, The cross section of expected stock returns, *Journal of Finance* 47, 427–465.

Fama, Eugene F., and James D. MacBeth, 1973, Risk, return, and equilibrium: Empirical tests, *Journal of Political Economy* 81, 607–636.

Foster, Douglas, Tom Smith, and Richard Whaley, 1997, Assessing goodness-of-fit of asset pricing models: The distribution of the maximal R-squared, *Journal of Finance* 52, 591–607.

Gallant, Ronald, Peter E. Rossi, and George Tauchen, 1992, Stock prices and volume, *Review of Financial Studies* 5, 199–242.

Lim, Terence, Andrew W. Lo, Jiang Wang, and Petr Adamek, 1998, Trading volume and the Mini-CRSP database: An introduction and user's guide, MIT Laboratory for Financial Engineering Working Paper No. LFE–1038–98.

Lo, Andrew W., and Craig MacKinlay, 1997, Maximizing predictability in the stock and bond markets, *Macroeconomic Dynamics* 1, 102–134.

Lo, Andrew W., and Jiang Wang, 2000, Trading Volume: Definitions, Data Analysis, and Implications of Portfolio Theory, *Review of Financial Studies* 13, 257–300.

Lo, Andrew W., and Jiang Wang, 2003, Trading Volume, in M. Dewatripont, L. Hansen, and S. Turnovsky, eds.: *Advances in Economics and Econometrics: Theory and Applications, Eighth World Congress, Volume II* (Cambridge University Press).

Lo Andrew W., and Jiang Wang, 2005, Trading volume: Implications from an intertemporal asset pricing model, working paper, MIT.

MacKinlay, Craig, 1987, On multivariate tests of the CAPM, *Journal of Financial Economics* 18, 341–371.

MacKinlay, Craig, 1994, Multi-factor models do not explain deviations from the CAPM, *Journal of Financial Economics* 38, 3–28.

Merton, Robert C., 1971, Optimal consumption and portfolio rules in a continuous-time model, *Journal of Economic Theory* 3, 373–413.

Merton, Robert C., 1973, An intertemporal capital asset pricing model, *Econometrica* 41, 867–887.
Pastor, Lubos, and Robert F. Stambaugh, 2002, Liquidity risk and expected stock returns, *Journal of Political Economy* 111, 642–85.
Tkac, Paula, 1996, A trading volume benchmark: Theory and evidence, working paper, Department of Finance and Business Economics, University of Notre Dame.
Wang, Jiang, 1994, A model of competitive stock trading volume, *Journal of Political Economy* 102, 127–168.