National Committee for Fluid Mechanics Films

FILM NOTES

for

ROTATING FLOWS*

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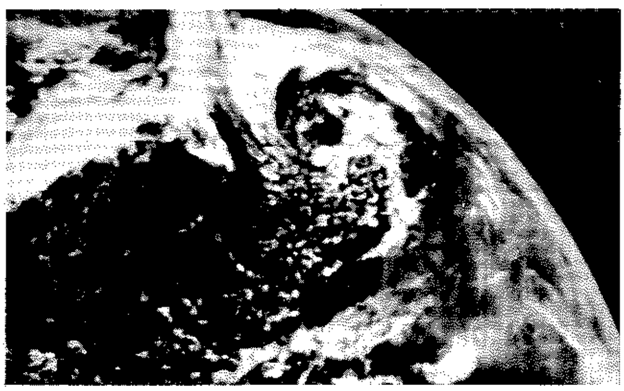
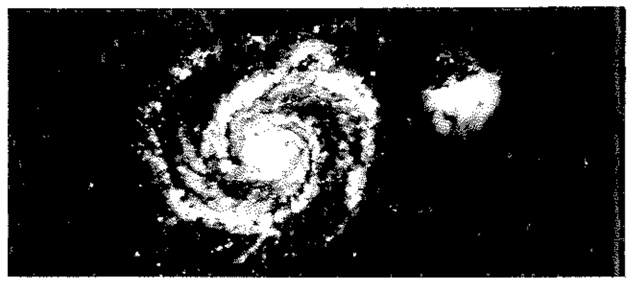
Introduction

Rotating fluids occur in a wide variety of technical contexts and in geophysics, particularly in the atmosphere and the oceans (Figs. 1 and 2). The phenomena involved are so varied that, in order to illustrate some of the principal features, we shall confine our attention to homogeneous fluids and to motions that do not deviate greatly from a rigid-body rotation. It is then advantageous to use a coordinate frame which rotates at an appropriate rate Ω with respect to inertial space. The inviscid fluid equations of motions take the form**



1. Cloud patterns seen from a satellite show rotational effects in the earth's atmosphere. (Courtesy V.E. Suomi, University of Wisconsin, and T. Fujita, University of Chicago.)



2. The whirlpool galaxy in Canes Venatici. (Courtesy Mt. Wilson and Palomar Observatories.)

$$\underbrace{\frac{\delta \mathbf{U}}{\delta t} + \mathbf{U}\cdot\nabla\mathbf{U}}_{\text{Relative acceleration}} = \underbrace{-\frac{1}{\rho}\nabla p}_{\text{Press. grad.}} \underbrace{- \nabla \phi_g}_{\text{Gravity}} \underbrace{- \nabla\left(\frac{\Omega^2 r^2}{2}\right)}_{\text{Centrifugal}} \underbrace{- 2\,\Omega\times\mathbf{U}}_{\text{Coriolis}}$$

where ϕ_g is the gravitational potential, the velocity vector $\mathbf{U}$ is measured relative to the rotating frame,

**The first four terms alone would constitute the complete equation of motion in a non-rotating, *inertial*, system. In the *non-inertial* rotating frame two additional terms are present.

**ROTATING FLOWS, a 16-mm color sound film, 29 minutes in length, was produced by Education Development Center under the direction of the National Committee for Fluid Mechanics Films, with the support of the National Science Foundation and the Office of Naval Research. Additional copies of the film notes and information on purchase and rental of the film may be obtained from the distributor:*

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and the other symbols are customary. The last two terms are parts of the absolute acceleration which are customarily shifted to the right-hand side and interpreted as forces. $-\nabla(\Omega^2 r^2/2)$ is the centrifugal force which, since it is a function of relative position only, can be combined with the gravity term to give an apparent gravitational force. $-2\,\Omega \times \mathbf{U}$ is the Coriolis force, which is responsible for many of the unfamiliar features of rotating flows. A careful discussion of the origin and status of these terms is given in Ref. (1).

The Rossby Number and the Frequency Ratio

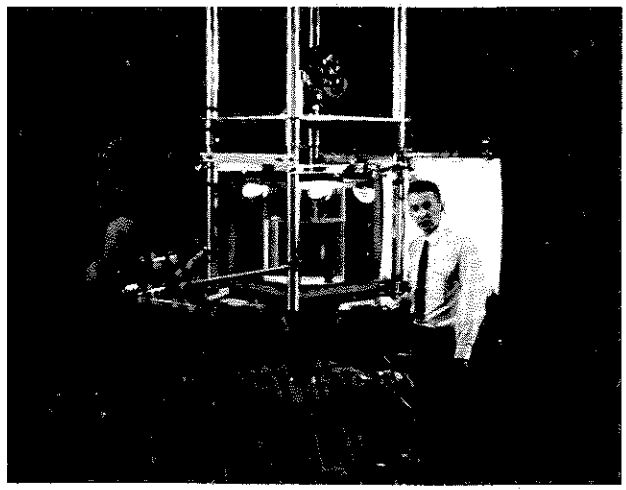
Two dimensionless parameters are especially important in the examples we shall consider — they characterize the relative importance of fluid relative accelerations as against Coriolis forces. If σ is a characteristic frequency, $\mathbf{U}$ a characteristic (relative) fluid velocity, and L a characteristic length, these parameters are the Rossby number, $U/L\Omega$, and a frequency ratio σ/Ω. The Rossby number has order of magnitude of the ratio of convective acceleration to the Coriolis force, $(|\mathbf{U}\cdot\nabla\mathbf{U}| \div |2\Omega\times\mathbf{U}|)$; and the frequency ratio has order of magnitude of the ratio of local acceleration to the Coriolis force, $\left(\left|\frac{\delta \mathbf{U}}{\delta t}\right| \div \left|2\Omega\times \mathbf{U}\right|\right)$.

If the Rossby number is much less than one, the relative convective acceleration is small, and if the frequency ratio is low, the local acceleration is small. Under these conditions there is a very close balance between the vertical component of the pressure gradient and apparent gravity (hydrostatic balance). More importantly, there is a close balance between the *horizontal component* of the pressure gradient and the horizontal component of the Coriolis force. Such flows are called geostrophic. They occur commonly in the atmosphere and oceans, where the Rossby number is small mainly because the scale of the motion is large. Full discussions of the theoretical features of geostrophic flows and of additional dimensionless parameters (such as those associated with the viscosity of real fluids) are given in Ref. (2).

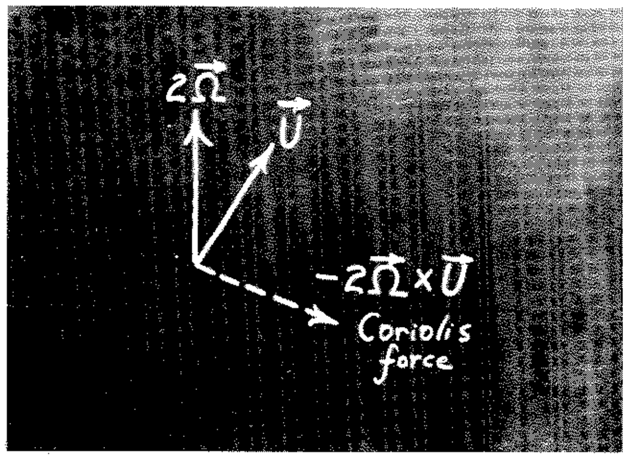
Elementary Coriolis Effects

The experiments in the film were carried out on a vertical-axis turntable driven by a very stable variable-speed motor and transmission (Fig. 3). The rotation axis must be vertical within a few seconds of arc when a free surface is present on the water. The top and side cameras rotated with the fluid containers, and all sequences where the rotation was not zero are identified by an indicator arrow. All rotations are counterclockwise seen from above.

One of the initial and subtle surprises is that direct



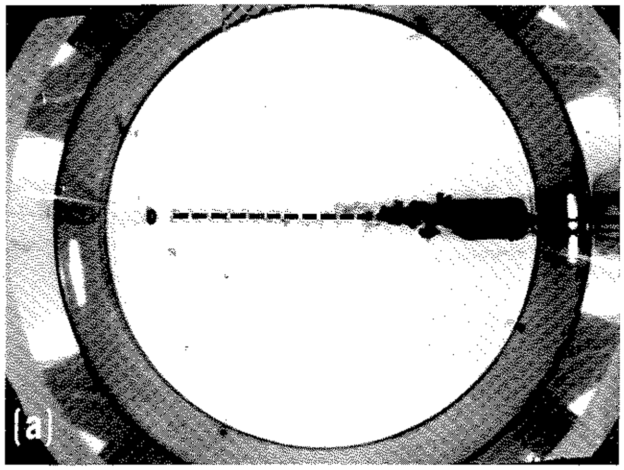
3. Rotating turntable carrying cylindrical tank inside a square tank to reduce optical refraction effects. Top and side cameras rotate with the tanks.



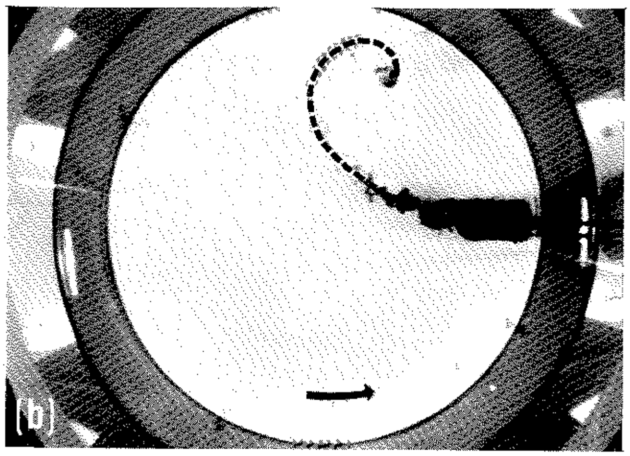
4. The Coriolis force is directed to the right of the rotational and relative velocity vectors.

reactions of the fluid to the Coriolis force (in the sense of motions deflecting to the right, toward $-2\Omega\times\mathbf{U}$) (Fig. 4) are obvious mainly for high Rossby numbers, where the Coriolis forces are relatively small. A small three-dimensional fluid body projected horizontally from a vortex-ring generator travels straight across a diameter of a cylindrical tank when the rotation is zero (Fig. 5a). In the same situation, but with the tank and fluid rotating, the vortex ring curves sharply to the right (Fig. 5b); the curvature increases as the horizontal velocity decreases. The Rossby number is of order one or more throughout (Ref. 3).

The effects of Coriolis forces are more complicated in an extended fluid body. When the rotation is zero, the lowest-frequency sloshing mode of surface gravity waves in a layer of fluid in a circular cylinder has the property that the free surface merely tilts back and forth (Fig. 6). Particles on the surface move horizontally in linear harmonic motion perpendicular to a nodal diameter. When the fluid layer is rotating (at

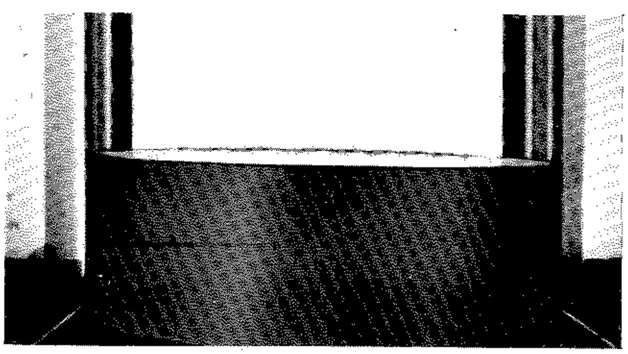


5a. A small parcel of dyed fluid, projected from a vortex-ring generator, travels straight across a non-rotating tank.

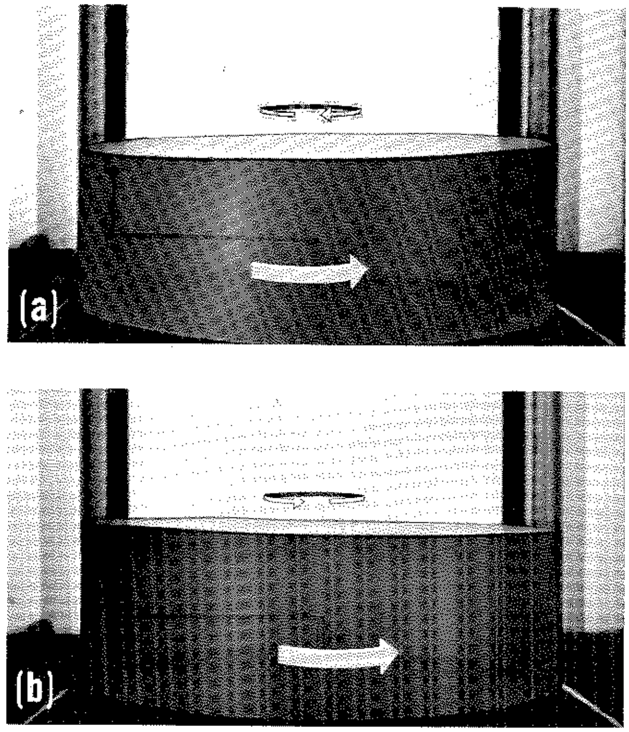


5b. When the tank and fluid are rotating counterclockwise at a moderate rate, the projected vortex ring curves sharply to the right (clockwise).

2.0 sec.$^{-1}$), there are two fundamental modes, now progressive, with crests moving either with, or opposed to, the sense of rotation (as seen in the rotating frame) (Fig. 7). These progressive modes stem basically from the transverse deflections due to the Coriolis forces, but the kinetic effects are substantially modified by the



6. Free surface tilt in the lowest sloshing mode of a gravity wave generated by a vertically-oscillating disk positioned near the rim on the left side. The tank is not rotating. Wave frequency = 8.4 sec.$^{-1}$.



7. When the fluid layer of Fig. 6 is rotating counterclockwise, a disk frequency 15% higher than the non-rotating fundamental mode produces a progressive wave which travels clockwise around the tank (a). A second progressive mode, at a frequency 13% lower than the non-rotating mode, travels counterclockwise (b).

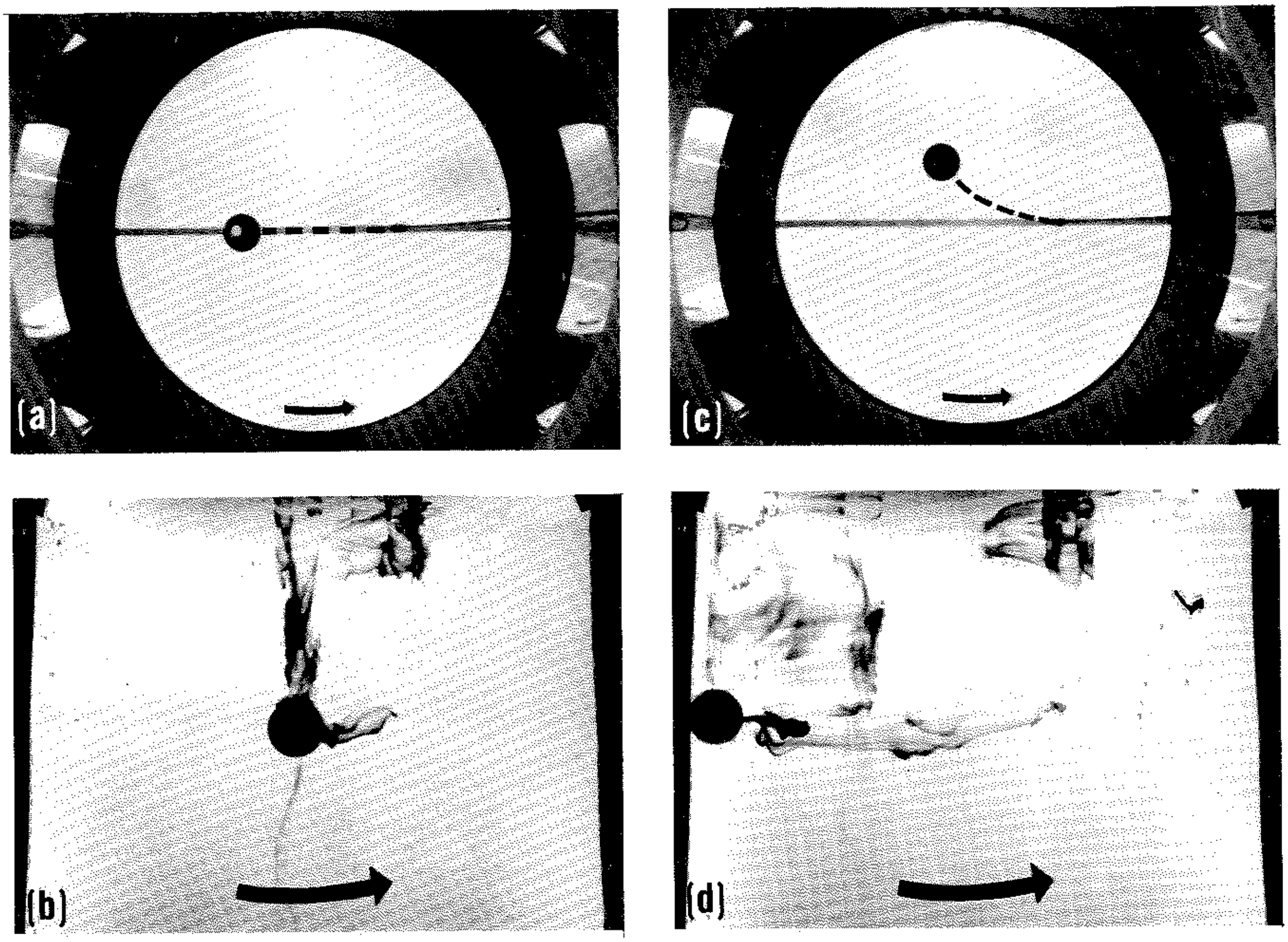
pressure fields. Thus the "negative" mode whose angular phase speed is clockwise has a 15% higher frequency than the non-rotating mode, while the "positive" mode has 13% lower frequency (Ref. 4). Moreover, while the horizontal trajectories of particles seen from above are circles curving to the right (in the same sense as the Coriolis force) for the negative mode, they are circles described to the left, exactly contrary to elementary expectation, for the positive mode.*

Low-Rossby-Number Motions Around Spheres

Some of the most striking instances of low-Rossby-number geostrophic flows were demonstrated by G. I. Taylor about 1920 (see Ref. 2 for discussion and sources), using various cases of flow around an immersed sphere. These and some later cases are considerably illuminated by thinking in terms of the vortex tubes of the absolute motion. In a low-Rossby-number, low-frequency-ratio, homogeneous flow, the inviscid Helmholtz vorticity equation contains (in the lowest-order approximation) only the term

$$(2\Omega \cdot \nabla)\mathbf{U} \approx 0.$$

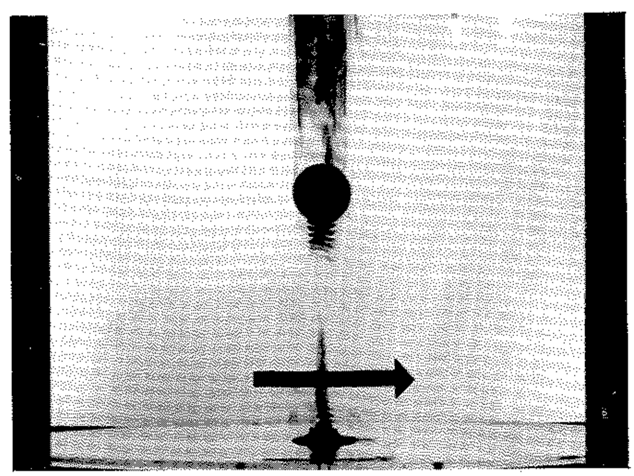
*The sequences in the film using aluminum powder on the top surface also show some unrelated motions resulting from local flow around the generating disk that is located below the free surface.



8. (a) A neutrally buoyant sphere, suspended from a long string so that it is free to deflect, travels straight across a diameter when pulled at a very slow speed in rotating fluid. (b) In a side view, dye streaks above the sphere show that a column of fluid is carried with the sphere at low Rossby number. (c) When the towing speed is increased so that the Rossby number is no longer small, the sphere deflects to the right. (d) A side view shows that the dyed column is left behind when the sphere is accelerated to higher Rossby number.

This is the Taylor-Proudman theorem (Ref. 2). It is equivalent to the statement that the relative velocity field does not vary in the direction of the rotation axis (since $2\Omega \cdot \nabla$ is a differentiation operation in that direction), and that the flow tends to be two-dimensional in planes perpendicular to the rotation axis. The absolute vortex tubes tend to remain parallel to the axial direction. They resist both bending and shrinking or stretching.

A small, nearly neutrally buoyant sphere was suspended on a fine thread and translated horizontally at a very slow speed. The sphere moved along a straight path, instead of deflecting to the right (Fig. 8a). It moved straight because a pressure field developed which exactly compensated the Coriolis force on the sphere. Dye trails above the sphere show that a Taylor column was carried with the sphere (Fig. 8b); in this manner the flow becomes approximately two-dimensional, in accord with the Taylor-Proudman theorem. When the towing speed is increased to a higher Rossby number, the flow becomes more three-dimensional, is no longer geostrophic, and the sphere deflects to the right (Fig. 8c). When the sphere is accelerated, the dye column is left behind (Fig. 8d).



9. A sphere is towed slowly parallel to the rotation axis. Ink shows the Taylor column above the sphere, and the helical motion in the column below the sphere where there is a strong counterclockwise spin.

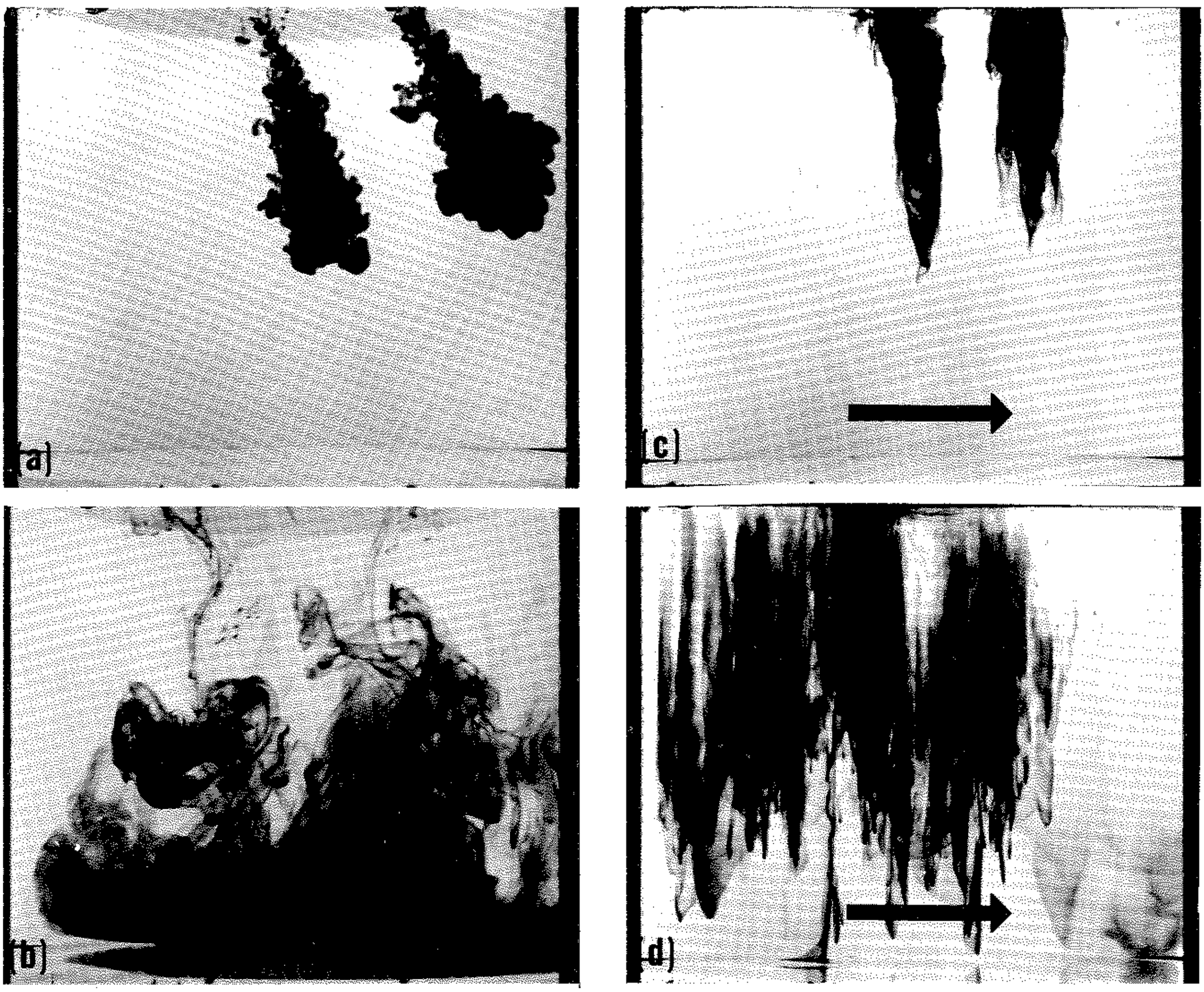
When the sphere is moved slowly upward parallel to the rotation axis, dye trails again show a Taylor column both above (forward wake) and below the sphere (Fig. 9). In the fluid ahead of the sphere, the compression of the absolute vortex tubes generates clockwise relative spin, while behind the sphere, the stretching generates counterclockwise relative spin. The corresponding pressure fields yield increased pressure in front and decreased pressure behind, so that the pressure drag on the sphere can be increased by several orders of magnitude. Thus the terminal velocity of a positively buoyant sphere is large when the rotation is zero, and becomes very small when the rotation is high enough to make the Rossby number small.

Taylor Walls

Even disorganized velocity fields can be radically affected by vortex tube stability when the Rossby number is small. Figures 10a and 10b show a turbulent field produced by injecting dyed water into a resting body of fluid. In Fig. 10c, the fluid was initially rotating rigidly and the dye was injected in an identical manner. The initial three-dimensional motions are similar to Fig. 10a, but they are rapidly converted (Fig. 10c) to a nearly two-dimensional flow as the velocities decay and the Rossby number decreases. The dye eventually becomes distributed in vertical Taylor walls (Fig. 10d). While the system is highly stable to the initial three-dimensional motions and rapidly converts them into two-dimensional geostrophic flows, it is nearly neutral to the latter, which undergo only a very slow viscous decay.

Inertia Oscillations and Rossby Waves

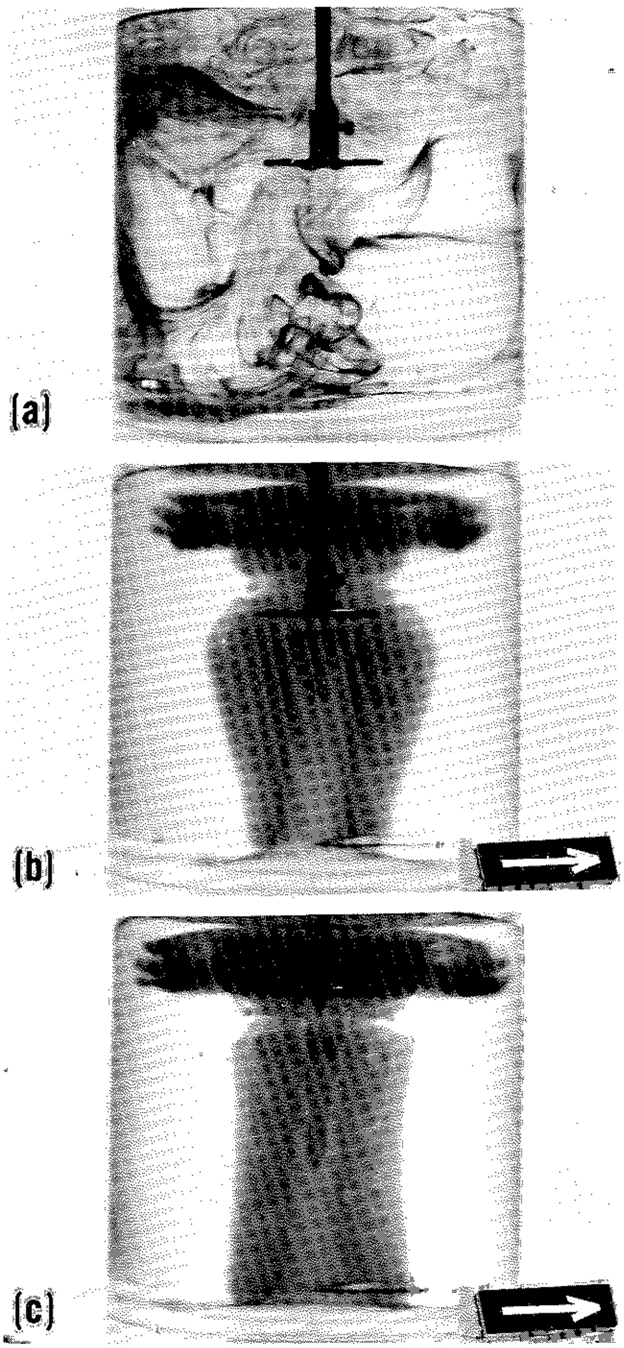
The rotational stiffness of the axial vortex tubes (or, alternatively, the resistance to changes of fluid circuit length in Kelvin's circulation theorem) make possible modes of oscillation that do not exist in non-rotating



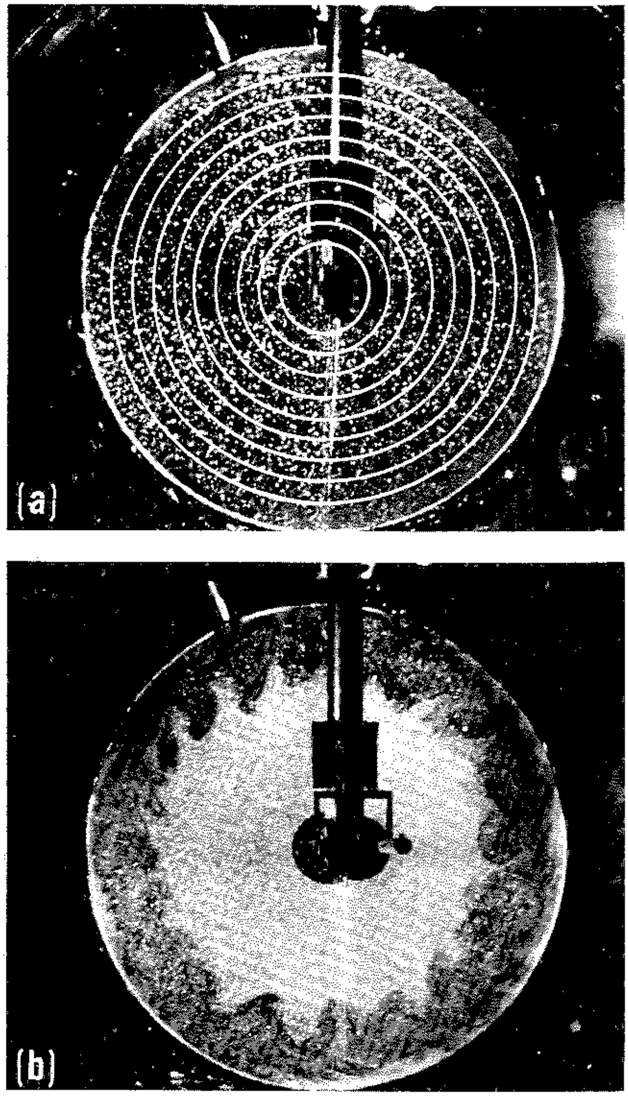
10. When dye is injected into non-rotating water (a), it spreads in a typical turbulent field (b). With identical ink injection into rotating water, the initial motions are rapidly converted to nearly two-dimensional motions (c), and after a time, the ink is found distributed in vertical Taylor walls (d).

fluid bodies. A small neutrally buoyant sphere, free to move on a vertical thread in rotating fluid, will oscillate up and down when given an impulse. This is a response to an inertia oscillation in the fluid.

In a rotating circular cylinder, normal modes of this type of oscillation can be generated by oscillating a small disk up and down along the axis at the proper frequency and position (Refs. 2 and 5). When there is no rotation, ink injected near the disk shows only a disorganized flow developing from vortex rings shed off the edge of the disk (Fig. 11a). With the cylinder rotating, the ink forms a stable Taylor column as the disk oscillates, in this case at a frequency ratio of 1.25. This frequency gives the mode with a nodal plane at middepth for the particular depth-radius ratio (2.00) of this cylinder. In contrast to a two-dimensional geostrophic flow, the oscillating ink column undergoes substantial expansions and contractions (and corresponding vorticity changes) as shown in Figs. 11b and 11c at two phases differing by half a period. Inertia oscillations of the same general character have been shown to be very common in the oceans and large lakes (Ref. 6).



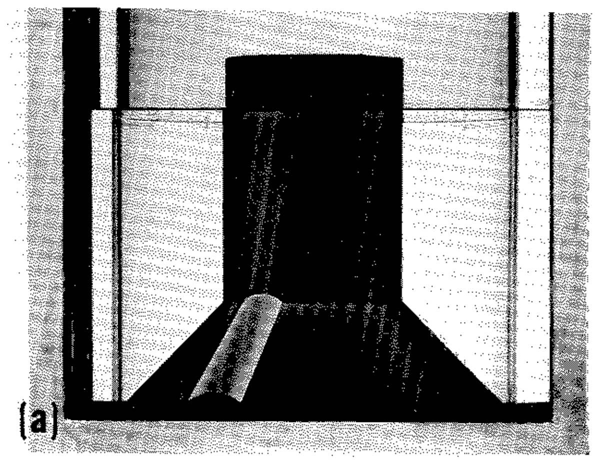
11. When there is no rotation, ink injected near a vertically oscillating disk shows only a disorganized flow (a). With rotation, the ink remains in a column which oscillates in an inertial mode with a nodal plane at the midpoint (b). (c) Ink column 180 degrees out-of-phase with (b).

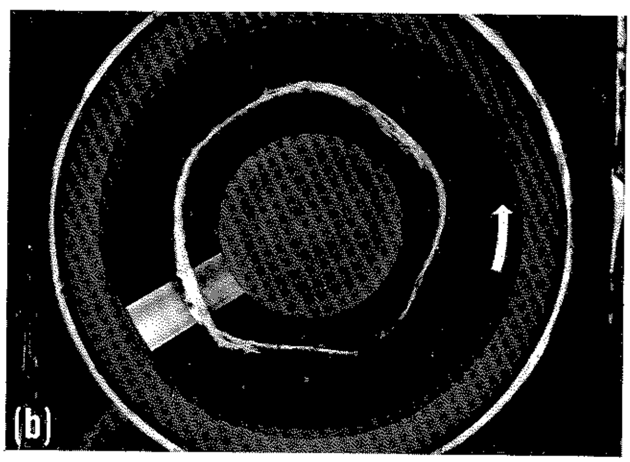


12. (a) A centrally placed disk oscillating vertically in a relatively shallow circular body of water produces rings (delineated by the overlaid lines) which oscillate in a tangential direction. The direction of motion is opposite for adjacent rings. (b) When the amplitude of the disk is increased beyond a certain critical value, the ink near the circumference shows that the motion becomes unstable.

In a much flatter circular body of water than that of Fig. 11, and with an excitation frequency ratio of only 1/2, the normal mode obtained is one with a large radial wave number (10 or so nodal circles), shown in plan view in Fig. 12a. An ink streak placed in the fluid along a circle within one cell of the normal mode expands, contracts, and simultaneously oscillates in the tangential direction during the inertia oscillation (Ref. 2, 5). When the amplitude of the generating disk is increased beyond a certain point, the flow becomes unstable to a regular set of waves in the azimuthal direction, Fig. 12b); these waves amplify and often become complete rows of oscillating vortices (Ref. 7).

A similar class of very low-frequency waves known as Rossby waves (Ref. 2) are very important in the motions of the atmosphere and oceans. This type of wave arises where there is a *variation* in the depth of a fluid such that axial vortex tubes are forced to shrink or stretch in a low-Rossby-number motion. Our example of a Rossby wave is produced in the annular cylinder shown in Fig. 13a. A conical bottom produces radial depth variation. A low, smooth radial mountain on the cone forces the wave perturbations when counterclockwise zonal flow is produced by slowly reducing the rotation of the container from an initial high value. With a Rossby number of about .045, a train of 5 sinusoidal stationary waves is produced (Fig. 13b). For larger characteristic values of the Rossby number, the stationary wave number decreases (Ref. 2).

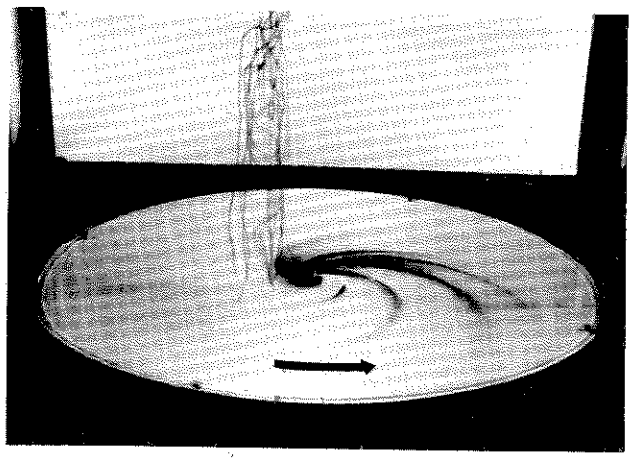




13. (a) Side view of annular cylinder combination with a conical bottom. The ratio of the outer to inner depth is about 5/3. (b) In plan view, a dye trace shows a train of 5 stationary Rossby waves excited in the annulus when there is tangential flow past a low ridge on the conical bottom. The dye trace was released at about middepth.

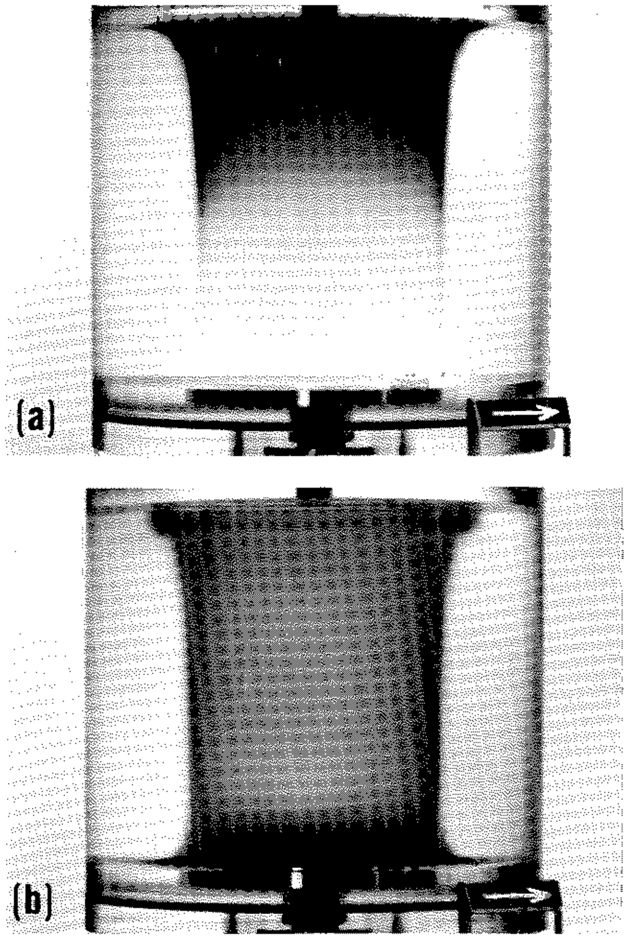
Ekman Layers and Free Shear Layers

Effects of viscosity can be very substantial in rotating systems. The Taylor-Proudman theorem deals with inviscid fluids, for example, and does not hold in boundary-layer regions for any Rossby number. Figure 14 shows one of the simplest types of rotating boundary layers, called an *Ekman layer* (Ref. 2). The water and tank were initially in rigid-body rotation, and the tank speed was then increased to a slightly higher rotation rate. The dye streaks in Fig. 14 show the strong outward spiraling motions in the Ekman layer; the fluid in the inviscid region above is in clockwise relative motion that is almost purely tangential. A downward flux over the central region is required to supply the radial flux in the Ekman layer. This stretches the vortex tubes in the interior, and has the effect of increasing the angular velocity of the fluid to the new value for the tank in a much shorter time than would be expected from ordinary viscous diffusion mechanisms.



14. Ekman boundary layer produced by increasing the rotation rate of the floor under a rotating cylinder of water. The vertical streaks left by falling dye crystals show that the main body of water continues to rotate. The spiral streaks on the floor show that only in a very thin boundary layer are there appreciable radial velocity components.

If the currents in the inviscid region vary from one place to another, the transverse Ekman layer fluxes also vary, requiring exchanges of fluid between the viscous and the inviscid regions that may be quite localized. In Fig. 15 the cylinder has a central disk



15. Side view of a cylindrical free shear layer moving downward from the solid lid of a rapidly rotating circular cylinder 130 sec.$^{-1}$) toward the edge of a more slowly rotating disk in the base. The Rossby number of the relative disk rotation is about 10^{-3} (a). (b) When the dye reaches the disk in the base, it flows inward along the disk and then upward in a slow flow parallel to the axis of rotation.

which is flush with the clear plastic bottom. The disk is rotated at a slightly lower speed than the cylinder. The interior fluid out to the disk radius adjusts to a rotation speed halfway between that of the disk and that of the upper lid. The accommodation to the no-slip boundary condition occurs in an Ekman layer on the disk, and in another Ekman layer under the lid. The fluid in the outside ring rotates at nearly the same speed as the cylinder. Ink is injected through a central hole in the solid lid, which is in contact with the fluid. The ink spreads rapidly outward in the top Ekman layer to the same radius as the disk, and stops there, even though the lid is a single rigid sheet. It then descends in a thin cylindrical Taylor sheet with a hollow core (Fig. 15a), a so-called free shear layer, to the base disk edge. There it flows inward in the disk Ekman layer and eventually completes the circuit by a slow upward flow parallel to the axis (Fig. 15b).

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