Part 1 21605
Part 2 21606

National Committee for Fluid Mechanics Films

FILM NOTES for VORTICITY*

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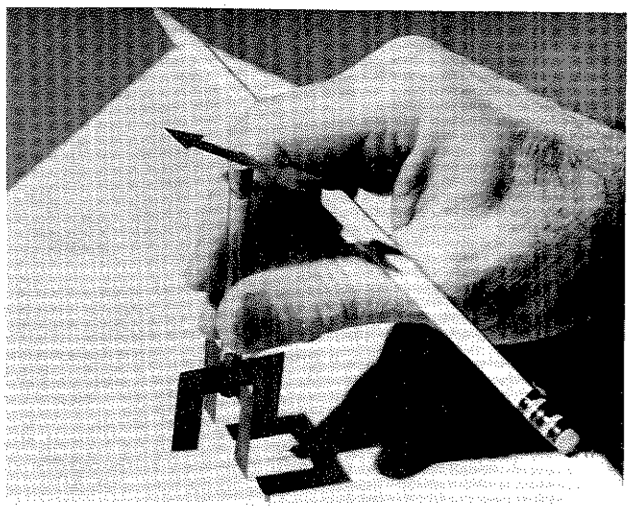
Introduction

The experiments in this film illustrate the concepts of vorticity and circulation, and show how these concepts can be useful in understanding fluid flows.

The vorticity is defined as the curl of the velocity vector: $\omega = \nabla \times \mathbf{V}$. Thus each point in the fluid has an associated vector vorticity, and the whole fluid space may be thought of as being threaded by vortex lines which are everywhere tangent to the local vorticity vector. These vortex lines represent the local axis of spin of the fluid particle at each point. In two dimensions, *the vorticity is the sum of the angular velocities* of any pair of mutually-perpendicular, infinitesimal fluid lines passing through the point in question. For rigid body rotation, every line perpendicular to the axis of rotation has the same angular velocity: therefore the vorticity is the same at every point, and is twice the angular velocity.

Vorticity is related to the moment of momentum of a small spherical fluid particle about its own center of mass. Given some very complicated motion of a liquid, suppose that it were possible — by magic — suddenly to freeze a small sphere of the liquid into a solid, while



1. Vorticity meter. The four vanes at the bottom are rigidly attached at right angles to the vertical glass tube. The arrow is fixed to the tube and rotates with approximately the average angular speed of the pair of mutually-perpendicular fluid lines which coincide with the vanes. Thus the rate of rotation of the arrow is approximately half the vertical component of vorticity of the lump of water in which the vanes are immersed. Note that since the vanes are rigidly connected, the float does not respond to shear deformation of the two fluid lines, but only to their average angular velocity.

VORTICITY*, a 16-mm B&W sound film, 44 minutes in length, was produced by Education Development Center (formerly Educational Services Incorporated) under the direction of the National Committee for Fluid Mechanics Films, with the support of the National Science Foundation. Additional copies of the film notes and information on purchase and rental of the film may be obtained from the distributor:

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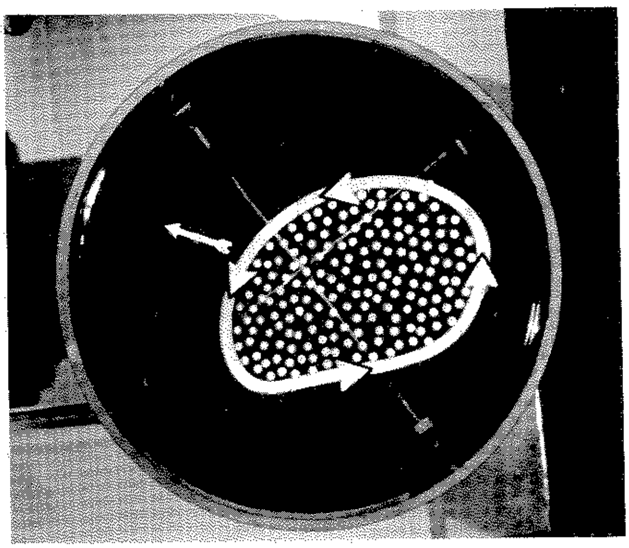
conserving the moment of momentum. The angular velocity of the solid sphere at the moment of its birth would be exactly half the vorticity of the fluid before freezing. Several dynamical theorems in effect relate the changes in vorticity of a fluid particle — and thus of its moment of momentum — to the moments of the forces acting on that fluid particle.

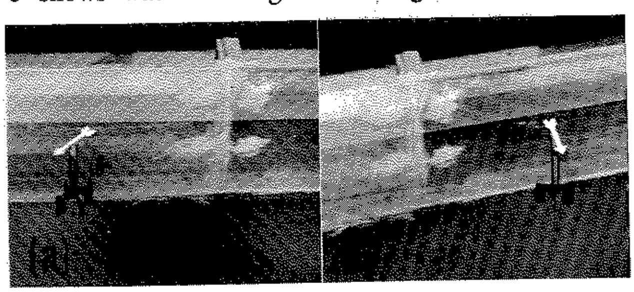
Fig. 1 shows a "vorticity meter" which floats in water with its axis vertical. Placed in a liquid which is in solid-body rotation (Fig. 2), the vorticity float



2. View from above of an open cylindrical tank containing water. It is mounted on a turntable which rotates about a vertical axis. The cross hairs are scribed on the bottom of the tank. After a long time the water is brought into solid-body rotation by viscous forces, and the vorticity float (arrow) moves as though it were rigidly affixed to the cross hairs. The drawing superimposed shows a closed curve *C* on which the circulation Γ is reckoned, and the associated vorticity flux through the area bounded by *C*.

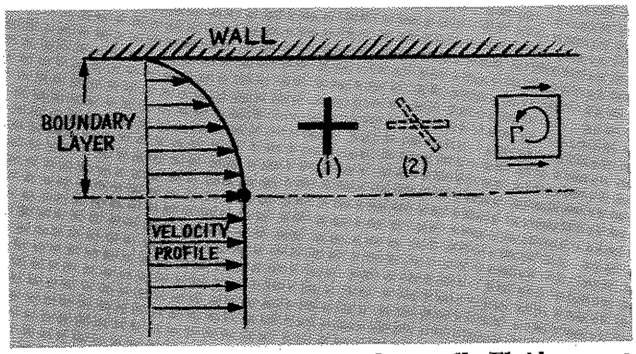
moves as though it were rigidly attached to the rotating tank. It has the angular velocity of the tank; the vertical component of vorticity at every point in the fluid is twice this angular velocity.

Sometimes the word "rotation" is used as a synonym for vorticity, but this does not mean that a flow has to be curved for vorticity to be present. For instance, Fig. 3 shows water flowing in a straight channel. The



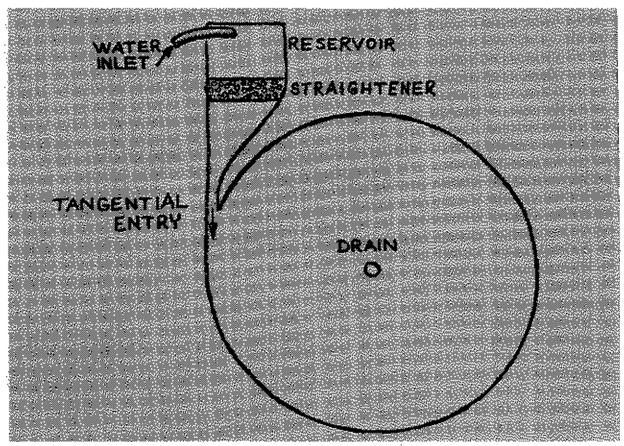
3. Water flowing from left to right in a channel with straight vertical walls. The vorticity meter is placed in the viscous boundary layer near one wall. As it moves downstream, the arrow turns counterclockwise. Picture (b) was taken a short time after picture (a).

streamlines are essentially straight and parallel to the side wall. But the rotation of the arrow shows that vertical vorticity is present. Near the wall is a viscous boundary layer in which the velocity increases with



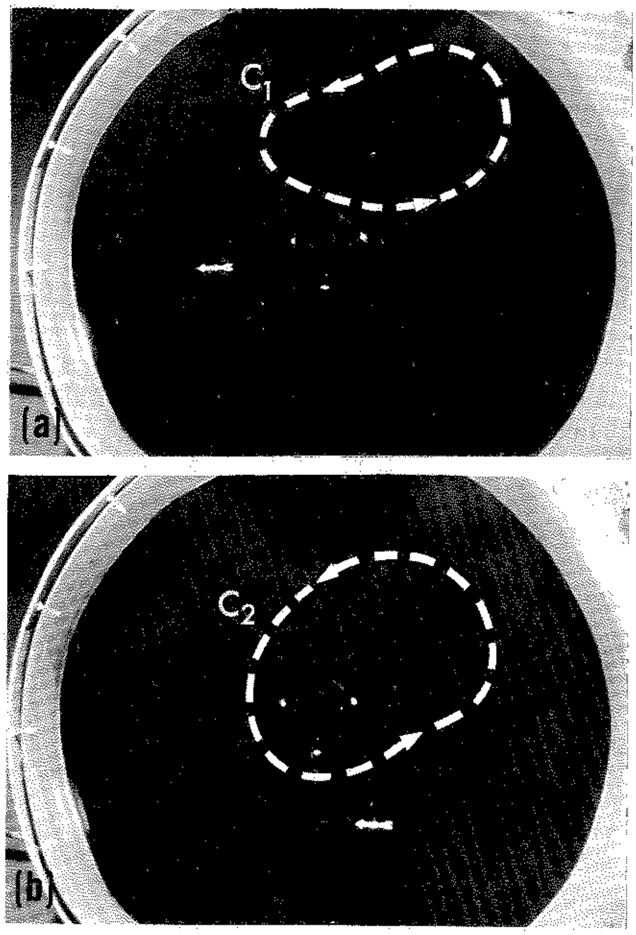
4. Viscous boundary layer near plane wall. Fluid cross at position 1 (solid lines) changes in form as it moves to position 2 (dashed lines). Square at right shows the closed circuit on which circulation is reckoned.

distance from the wall (Fig. 4). Examine the fluid cross. One leg moves downstream parallel to the wall while the other leg rotates counterclockwise owing to the non-uniform velocity distribution. Thus there is a net vorticity, and the vorticity meter of Fig. 3 turns counterclockwise.



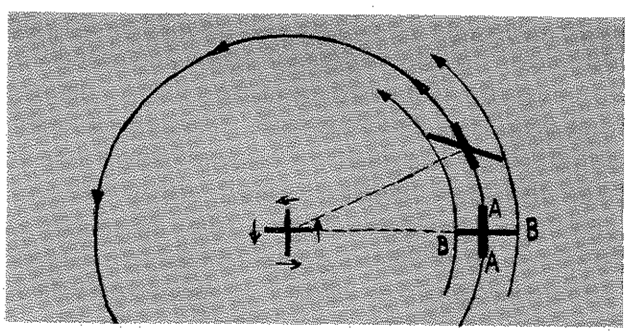
5. Plan view of sink-vortex tank. Water from the reservoir passes through the flow straightener, and thence into the tank through the tangential entry. After spiralling round and round in a very tight vortex, the water leaves vertically through the drain at the center.

On the other hand, the flow may be without rotation even though the streamlines are curved. Fig. 5 shows in plan view a tank for producing a sink vortex in which the streamlines are tight spirals and nearly circular. As shown in Fig. 6, the vorticity meter moves in a circular path but does not rotate. It moves in pure translation — as would a compass needle on a phonograph turntable. Consider a fluid cross at a point on a circular streamline (Fig. 7). Leg *A* follows the streamline, hence it rotates counterclockwise. Since the angular momentum of the fluid is conserved as it flows toward the drain, the tangential velocity varies



6. **Plan view of vorticity meter in the sink-vortex tank. Picture (b) was taken a short time after picture (a); the arrow direction has not changed. The drawings superposed show closed curves, C_1 and C_2, on which the circulation is reckoned for circuits which respectively do not enclose, and do enclose, the center of the vortex.**

inversely with the radius. Thus the velocity of the inner part of leg *B* is greater than the velocity of the outer part, and leg *B* turns clockwise. The clockwise turning rate of *B* is just equal and opposite to the counterclockwise turning rate of *A*. Hence the vorticity is zero. The vorticity meter, in averaging the rotations of legs *A* and *B*, translates, without rotation, on a circular trajectory.



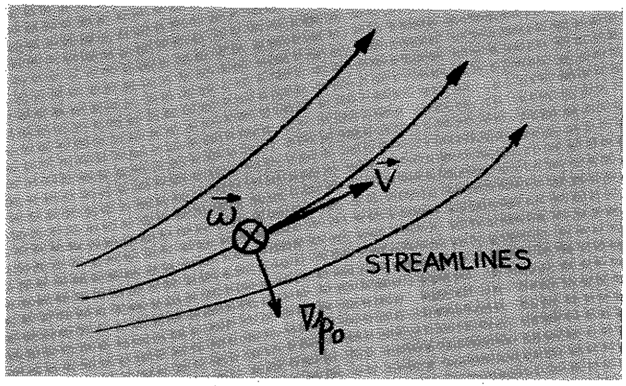
7. **Fluid crosses in a two-dimensional vortex. One is on a circular streamline, the other at the center.**

Crocco's Theorem

For the special case of steady motion of an incompressible, inviscid fluid acted on by conservative body forces, *Crocco's theorem* has the form

$$\mathbf{V} \times \omega = \frac{1}{\rho} \nabla p_o ; p_o \equiv p + \tfrac{1}{2} \rho V^2 + \rho U \quad (1)$$

where $\mathbf{V}$ is the vector velocity, ω the vector vorticity, and ρ the density. The stagnation pressure p_o is the sum of the static pressure p, the dynamic pressure $\rho V^2/2$, and the potential energy per unit volume ρU associated with the conservative body-force field.

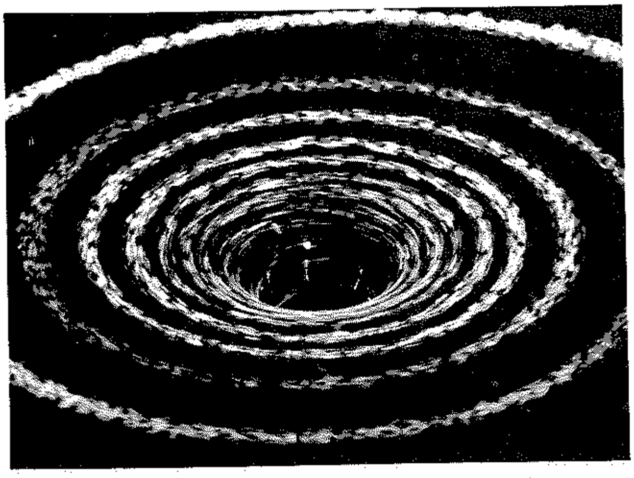


8. **In a two-dimensional flow, $\mathbf{V} \times \omega$ is in the plane of the flow and perpendicular to the streamlines.**

When a flow is two-dimensional in the plane of the paper, the vorticity vector is normal to the paper while the velocity vector lies in the paper and along the streamline (Fig. 8). By Crocco's theorem, the gradient of stagnation pressure is normal to both the velocity vector and the vorticity vector; thus it lies in the plane of the paper and normal to $\mathbf{V}$. Consequently the stagnation pressure, p_o, is constant along each streamline and varies between streamlines only if vorticity is present.

To illustrate, consider again the straight boundary layer of Figs. 3 and 4. The static pressure is uniform across the boundary layer but the velocity is variable. Thus the stagnation pressure is variable, and, by Eq. 1, vorticity is present. The velocity gradient is strongest near the wall and so is the gradient of stagnation pressure. When the vorticity meter is near the wall, the rate of spin is relatively large. With the vorticity meter farther out in the boundary layer, the rate of spin is smaller.

When the vorticity is zero, as in the sink-vortex tank (Figs. 5 and 6), Crocco's theorem says that the stagnation pressure must be everywhere the same. The spiral of the vortex is so tight (Fig. 9) that it is not much of a liberty to think of the streamlines as being concentric circles. One may verify that the uniformity of angular momentum, $Vr =$ const., is the condition for, (1) the stagnation pressure to be constant throughout, and (2) the free surface to be a hyperboloid



9. **A streamline of the sink vortex. Note hyperboloidal depression in center, over the drain.**

of revolution. For an inviscid fluid, the hole in the core would extend downward indefinitely. However, the high-velocity gradients and strain rates near the axis produce large viscous forces which reduce the depression to a deep dimple having a bottom.

A flow which is otherwise without rotation may contain small regions where the vorticity is very large. In the sink-vortex flow, for instance, the vorticity is generally zero (Fig. 6), except for a highly-concentrated core of vortical fluid right at the center. When the vorticity float finally drifts into the center, its motion, which hitherto was purely translational, is immediately changed to a pure and rapid rotation. Only at the singular point in the center of Fig. 7 do both arms of a fluid cross rotate in the same direction and thus produce a net vorticity.

Fluid Circulation

The fluid *circulation* Γ is defined as *the line integral of the velocity* **V** *around any closed curve* C. The circulation theorem — which is purely geometrical — equates the circulation Γ around C to the flux of the vorticity vector ω, through any surface area bounded by *C*.

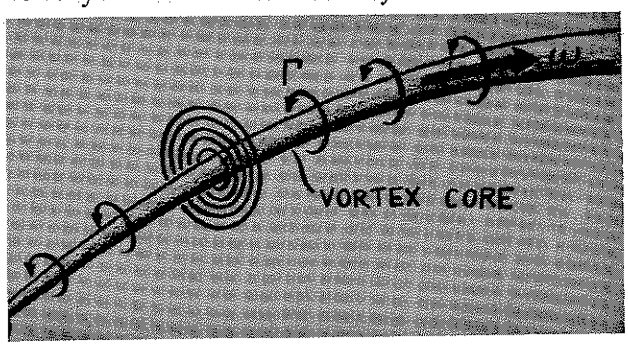
$$\Gamma = \oint \mathbf{V} \cdot \mathbf{dr} = \iint \nabla \times \mathbf{V} \cdot d\mathbf{A} = \iint \omega \cdot d\mathbf{A} \qquad (2)$$

If there is a definite circulation around C, then the fluid lying in any surface bounded by C must have vorticity. When the circulation is zero for *every* curve in a certain region, the fluid in that region must be entirely free of vorticity: the motion is then called *irrotational.*

Returning to the boundary-layer flow of Figs. 3 and 4, consider the circulation for the small square circuit in Fig. 4. Because of the non-uniform distribution of speed, there is a net circulation which, by Eq. 2, is related to the vertical vorticity of the enclosed fluid.

Vorticity may be distributed throughout the entire fluid. But often the vorticity is very large only in a thin thread of fluid while the remaining fluid is virtually without vorticity. Then we can simplify our thinking by lumping all the vorticity into a concentrated vortex line around which the fluid spins (Fig. 10), and by pretending that the remaining fluid is entirely free of vorticity. The finite amount of circulation around the core requires that the vorticity be infinite in the vortex line, which has zero cross-sectional area. In cross section, a straight vortex line with non-vortical fluid outside would appear as a point around which the fluid moves in concentric circles, the circumferential velocity varying inversely with radius.

In the solid-body rotation tank (Fig. 2), the vertical vorticity is everywhere equal to twice the angular velocity of the tank. Every horizontal circuit therefore has a circulation equal to twice the product of the angular velocity and the area bounded by the circuit.



10. **Schematic of a vortex core of strength Γ imbedded in otherwise irrotational fluid.**

In the sink-vortex tank the flow is non-vortical except for the concentrated vortex core which accounts for the whole circulatory motion. All fluid circuits not surrounding the core (Fig. 6a) have zero circulation because they contain no vorticity flux. All fluid circuits surrounding the core (Fig. 6b) have the same circulation because they contain the entire vorticity flux.

A wing generates lift because of the higher pressure below and the lower pressure above. According to Bernoulli's integral, the velocity on the upper surface must be greater than the velocity on the lower surface. This means that there is a net circulation around a lifting wing. Often we model this circulation as being produced by a fictitious vortex which is "bound" in the wing and which accounts for the circulatory movement (Fig. 19). The vorticity is really present, but it is distributed throughout the viscous boundary layer rather than concentrated in a single vortex line.

Kelvin's Theorem

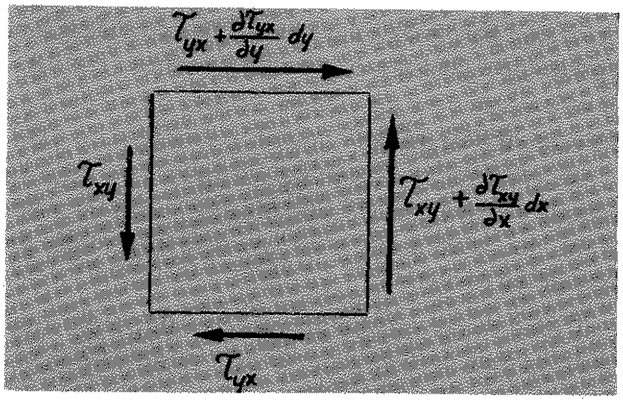
The concept of circulation is important mainly because of a powerful theorem evolved by Lord Kelvin from the dynamical laws of motion. It shows how the time rate of change of circulation Γ_c associated with a closed curve C always made up of the same fluid particles is governed by the torques produced by all the forces acting in the fluid:

$$\frac{D\Gamma_c}{Dt} = -\oint \frac{dp}{\rho} + \oint \mathbf{G} \cdot d\mathbf{r} + \oint \frac{\mu}{\rho} \nabla^2 \mathbf{V} \cdot d\mathbf{r} \quad (3)$$

The three terms on the right represent torques due to pressure forces, body forces, and viscous forces, respectively.

Viscous Torques

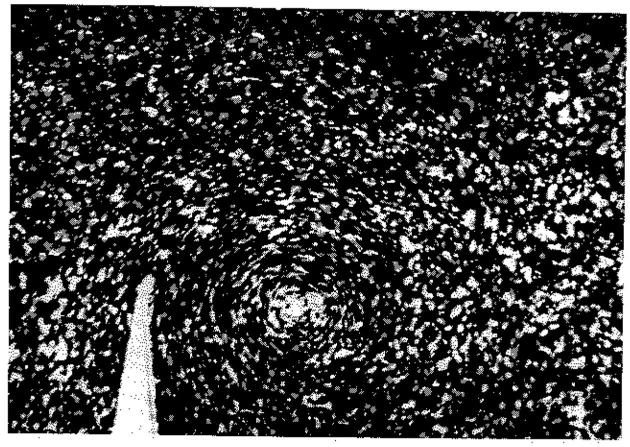
Let us consider first the torques produced by viscous forces acting on a fluid particle. A force diagram for a fluid particle (Fig. 11) shows that viscous forces are



11. Viscous stresses on a fluid particle.

indeed capable of producing torques about the center of mass. Such viscous torques change the vorticity of the fluid particle and thus the circulation on a bounding circuit. For instance, in the straight channel of Fig. 3, a vorticity float inserted just outside the boundary layer moves downstream for a while without turning. But vorticity is diffusing outward from the wall, and eventually the fluid in which the float travels reaches a position where a viscous torque produces vorticity. Then the float begins to spin.

When the water in the cylindrical tank illustrated in Fig. 2 is at rest, $\Gamma = 0$ for all circuits. When the turntable is started, the water in the middle does not at first move, because none of the forces which create circulation come into play there for a while. Next to the wall, the fluid moves promptly because of viscous stresses. These viscous forces (aided by outward flow in the boundary layer on the bottom of the tank) gradually accelerate fluid farther out from the wall and more and more of the fluid moves. In the end, viscosity brings all the fluid into a perfect solid-body rotation. At this limit, paradoxically, viscous forces have vanished altogether and there is nothing to force further changes in circulation. If the turntable is then stopped, the fluid continues to rotate as a solid body except near the walls of the tank where the viscous forces are large. Viscosity again causes the vorticity change to diffuse inward from the wall, decreasing the circulation more



12. Fluid flows from left to right past a sharp edge. The photo shows conditions soon after the flow has started impulsively. The flow separates behind the edge, and fluid in the surface of discontinuity, in which strong viscous forces act, forms a starting vortex which moves downstream. (After Prandtl.)

and more until finally the whole fluid is brought to rest.

When a fluid flows around a sharp edge (Figs. 12 and 23b), viscous and pressure forces in the boundary layer lead to a separated flow. The fluid which has been affected by viscous forces forms a concentrated vortex.

In Fig. 23b, two vortices are made by pulling a plate with sharp edges through the water. We can make the vortex visible in this experiment by placing a ping-pong ball in the dimple of the vortex. It remains there stably. In a channel of constant width and depth, the spin of the ball decreases with time because viscosity diffuses the vorticity of the core into the surrounding fluid.

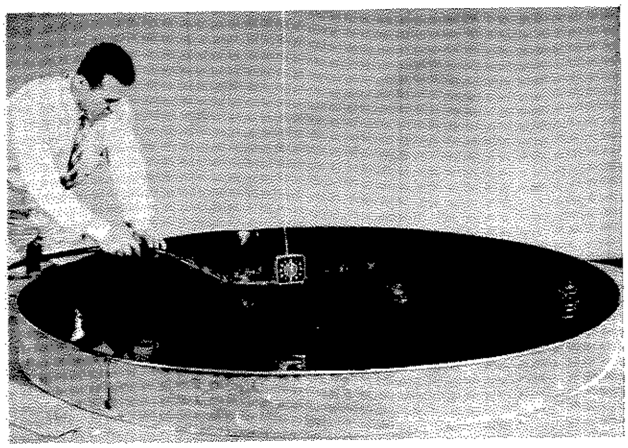
Body-Force Torques

If the body force **G** is *irrotational* (i.e. curl $\mathbf{G} = 0$), that is, *conservative,* the body-force term in Eq. 3 is zero. But, for *rotational* body forces, that is, *nonconservative* forces, this term is, in general, not zero.

Whenever the net body force **G** passes through the center of mass of a small sphere of fluid, it produces no torque to change the circulation. Centrally-directed forces like gravity are of this type. They are irrotational.

There are two important rotational body forces in fluid mechanics which can change circulation: (1) *Coriolis forces,* $(-2\,\Omega \times \mathbf{V})$, in rotating reference frames, and (2) *Lorentz forces,* $(\mathbf{J} \times \mathbf{B})$, due to the flow of an electric current at an angle to a magnetic field. In both these cases the line of action of the resultant body force need not go through the center of mass of a spherical particle. Because of such forces the oceans and the atmosphere are full of vorticity, as are magnetohydrodynamic flows.

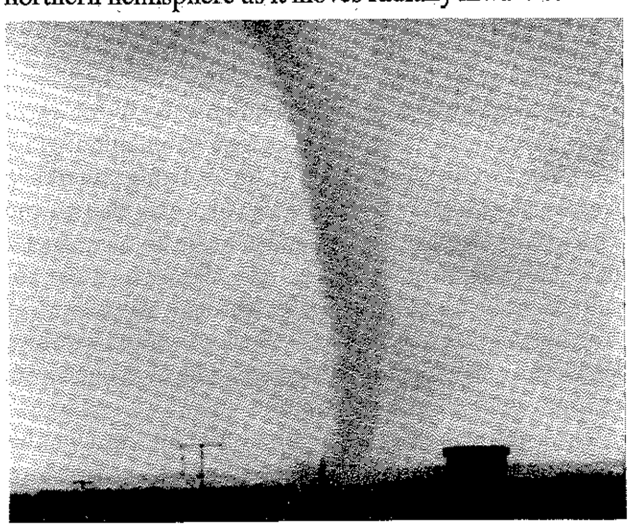
Does the vortex in the bathtub always turn in the same direction? Does it depend on which hemisphere you are in? You can't really tell in the bathtub, because the Coriolis force due to the earth's rotation, for



13. A tank six feet in diameter and six inches high, with a drain hole 3/8 inch in diameter at the center, is filled with water swirling *clockwise*. It is then covered to minimize motions induced by air currents, by buoyancy, and by impurities on the surface causing non-uniform surface tension, and it is allowed to stand for 24 hours. The flow is started by pulling a plug from the end of a hose, several feet long, attached to the drain. The experiment is carried out at latitude 42° N near Boston, Mass.

a speed toward the drain of about 0.2 inches per minute, is only about a billionth the force of gravity! Other effects all too easily mask that of the earth's rotation. However, with care, one can do an experiment dominated by the earth's Coriolis force (Fig. 13). Immediately after starting the flow, a small vorticity float with its vanes entirely submerged is placed over the drain. For the first 10 or 15 minutes there is no perceptible rotation of the float. But at about 15 minutes a distinct counterclockwise motion begins. At 24 minutes, with the tank nearly empty, the float is turning at about 0.3 rev./sec. This represents a 30,000-fold amplification of the earth's rotation at Boston. The reader can verify that this agrees, in order of magnitude, with the angular momentum being conserved.

The Coriolis force acting on a fluid particle in the northern hemisphere as it moves radially inward toward the drain is circumferential and counterclockwise. This force integrated around a circle contributes a counterclockwise torque in Kelvin's theorem. This tends to make the circulation increase counterclockwise with time. In the reference frame of the earth a fluid circle which starts at one radius with zero circulation therefore acquires counterclockwise circulation as time proceeds.



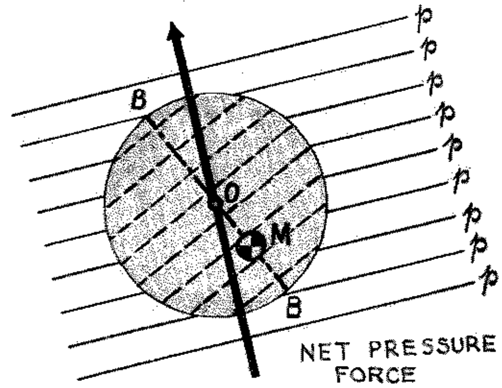
14. Funnel of a tornado.

Although the earth's Coriolis forces are small compared with gravity, they are extremely important to our everyday weather. They can also generate hurricanes. If the conditions of temperature and humidity are such that there is a strong local up-draft in some region, air must rush in from the sides to the "sink" forming the up-draft. This is like the water-tank experiment, but upside down, and a strong vortex is formed (Fig. 14).

Pressure Torques

When the fluid is effectively incompressible, or more generally when the density depends upon pressure alone, the term $\oint dp/\rho$ is zero, and thus does not change the circulation.

To see the physical significance of this term, consider the fluid particle in the circular region of Fig. 15. The lines of constant pressure (isobars) are shown by the solid lines. The pressure forces acting on the particle parallel to these lines exactly cancel each other, hence the net pressure force on the particle is perpendicular to the isobars and passes through the geometric center. The dashed lines represent the contours of constant density (isochors) in the fluid particle. If the isochors are parallel to the isobars — a situation described as *barotropic*, which means that ρ is a function of p alone — the line of action of the net pressure force goes through the mass center M, and produces no torque about M. But, if the isochors are *not* parallel to the isobars (Fig. 15), the net pressure force pro-



15. A small spherical fluid particle in a region where the isobars are the solid lines marked *p*. The net pressure force on the particle is perpendicular to the isobars and passes through the geometric center *O*. The dashed lines represent the isochors in the fluid. The center of mass *M* lies on the line *BB*, which is perpendicular to the isochors and passes through *O*.

duces a torque about M, and acts to change the circulation.

If the fluid is at rest in the gravity field of the earth, the isobars are horizontal. Since the circulation in a stationary fluid is forever zero, the surfaces of constant density must coincide with the surfaces of constant pressure. This is why the free surface of water in a pail is horizontal. If the pail contains oil floating on water, the interface is also horizontal. When we tilt the free surface and the interface by tipping the pail, the surfaces do not remain tilted: the isobars and isochors are now misaligned and the term $\oint dp/\rho$ in Eq. 3 creates a circulation which tends to make the surfaces horizontal again.

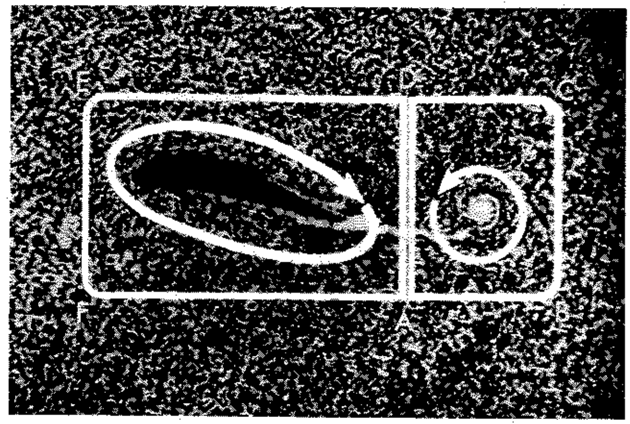
The circulations arising in natural convection systems are driven by the pressure-density term in Kelvin's theorem. One example is the hot-water or hot-air heating system in a house. Another is the principal circulation of the earth's atmosphere between the cold regions at the poles and the hot regions at the equator.

Origin of Irrotational Flow

Kelvin's theorem shows how irrotational flows may arise. Consider a motion which began from a state of rest. With no motion, there is no vorticity, and with no vorticity, there is no circulation. Suppose that the fluid is barotropic, that body forces if present are conservative, and that viscous forces are negligible. Then the circulation must forever remain zero on every fluid circuit, and the vorticity must also everywhere and forever remain zero.

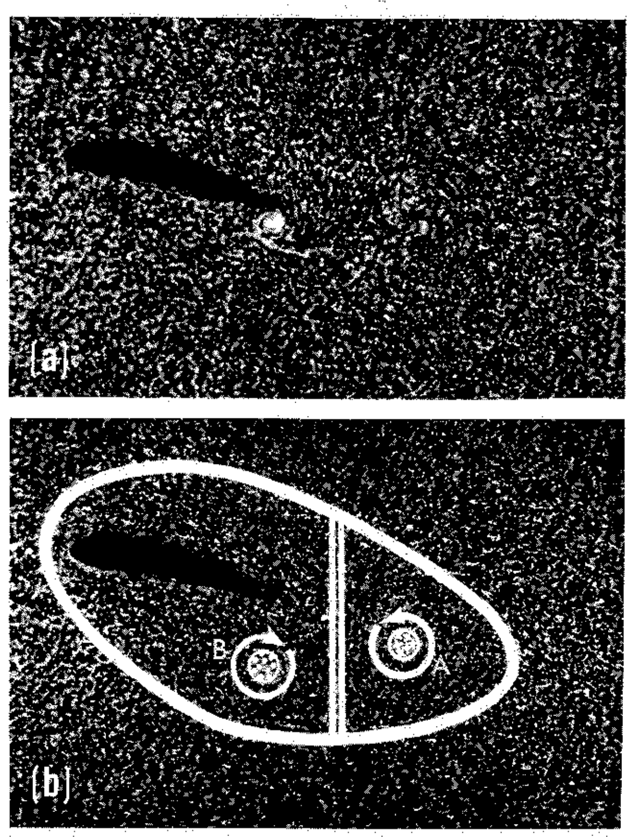
Let an airfoil begin to move suddenly through a fluid initially at rest. In the absence of viscosity, the circulation around any arbitrary fluid element is zero to begin with and therefore remains zero; thus the flow remains everywhere irrotational. There can be no circulation around the airfoil, and hence no lift. Fortunately, viscous friction, no matter how small, together with the no-slip condition at the solid surface, make lift generation possible. When the airfoil begins to move, viscous effects near the trailing edge result in the shedding of a so-called "starting vortex" (Fig. 16), and thus to a circulation on the curve *ABCDA*. But the fluid along the larger curve *ABCDEFA* is not subject to viscous forces, being outside the viscous boundary layer and wake. By Kelvin's theorem, the circulation on this curve must be zero. For this to be true there must be along the curve *ADEFA* surrounding the airfoil a circulation equal and opposite to that on the curve *ABCDA* surrounding the vortex. This circulation around the airfoil may be ascribed to a fictitious vortex bound in the airfoil, and is necessary for the production of lift.

When the airfoil stops, the bound vortex is shed as



16. An airfoil impulsively started from right to left sheds a "starting vortex" at the sharp trailing edge. (After Prandtl.)

a stopping vortex (Fig. 17), again because of viscous forces at the sharp trailing edge. On a circuit around either vortex, viscosity has acted and circulation is present. But on a circuit enclosing both vortices, and passing through fluid on which friction has never acted, the circulation remains zero. The equal and opposite vortices produce zero net flux of vorticity through any area containing both vortices.



17. Shortly after the airfoil of Fig. 16 is impulsively started, leaving a *starting vortex* "A", it is impulsively stopped, and sheds the *stopping vortex* "B" of opposite sign to "A" but of equal strength (after Prandtl). (a) Immediately after stopping. (b) A short time later.

Helmholtz's Vortex Laws

When all the torque-producing factors in Kelvin's theorem are absent, fluid dynamics can be given a beautiful geometrical interpretation in terms of Helmholtz's laws:

(1) Vortex lines never end in the fluid. They either form closed loops or end at a fluid boundary, and the circulation is the same for every contour enclosing the vortex line.

(2) A fluid line which at any instant of time coincides with a vortex line will coincide with a vortex line forever. (The vortex lines are, as it were, frozen to the fluid.)

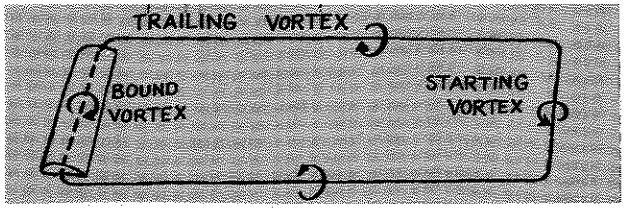
(3) On a vortex line of fixed identity, the ratio of the vorticity to the product of the fluid density with the length of the line remains constant as time proceeds ($\omega/\rho l = \text{const.}$). Thus, if the vortex line is stretched, the vorticity increases.

The vortices A and B of Fig. 17b are of equal and opposite strength. They move downward together, because the velocity field of B displaces the fluid at A, and vice versa, and because each vortex core is convected with the fluid to which it is frozen.

18. Smoke ring.

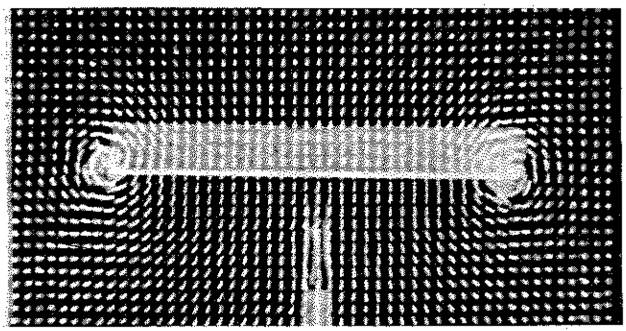
One can never see a smoke ring which is broken somewhere, because vortex lines can never end (Fig. 18). The fact that the smoke ring propels itself shows that the vorticity is frozen in the fluid: each element of fluid is propelled forward by the induced velocity fields of all the other elements of the ring vortex, and the whole vortex is thus convected by itself. The smoke, which marks the fluid, is carried with the vortex core, showing that the vorticity is locked to the fluid. When a smoke ring approaches a wall normal to the axis of the ring, it spreads out and slows down. This may be explained in terms of the induced velocity field of the fictitious image vortex on the other side of the wall, which, in effect, takes the place of the wall itself.

Fig. 19 shows the vortex system for a wing of finite span The circulation required to produce lift may be considered as originating from vorticity bound in the wing. But the bound vortex lines cannot end at the wing tips; they form vortex loops, closed by trailing



19. The vortices associated with a wing of finite span.

vortices and the starting vortex left at the airport. The trailing vortices from the tips of the lifting wing are made visible in Fig. 20. As the angle of attack of the wing is increased, the tip vortices grow in strength as the lift, the circulation, and the strength of the bound vortex also increase.

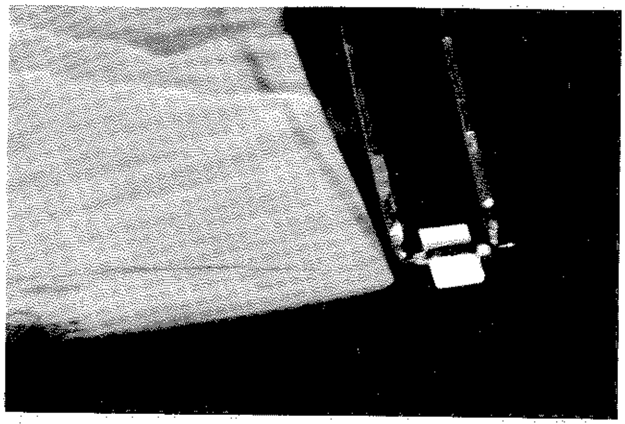


20. A view looking upstream along the axis of a wind tunnel. Between the camera and the trailing edge of the wing is a rectangular grid of fine wires, in a vertical plane, with wool tufts attached at the net points. The wool tufts align themselves with the flow, which is more or less perpendicular to the paper. The relatively heavy, horizontal white line is the trailing edge of the wing, and the dim white region above is the upper surface at incidence. One sees the projections of the wool tufts in a plane transverse to the direction of free flow and downstream of the trailing edge, and thus obtains an impression of the transverse velocity field. (Courtesy NASA.)

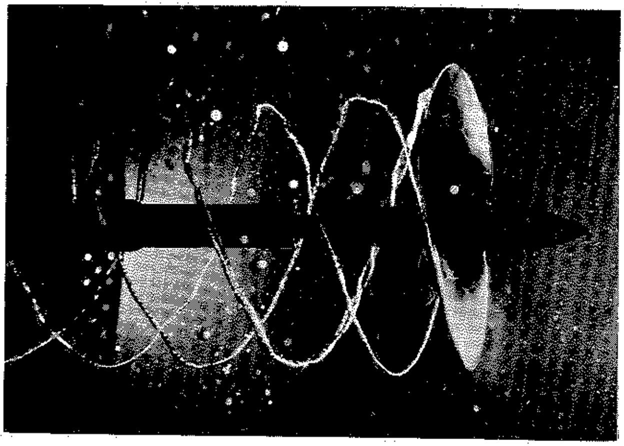
The little vorticity meter of Fig. 21 shows streamwise vorticity. Put right behind the wing tip, it spins very fast; if we move it slightly inboard or outboard of the tip, it hardly turns. The trailing vorticity is strongly concentrated at the wing tip.

The concentration of trailing vorticity in strong tip vortices results in very low pressures in the center of these vortices. Behind marine propellers, the water in the vortex cores may boil (cavitate), and this makes the trailing vortices from the blades easily visible (Fig. 22). They form a helical pattern.

The induced velocity field of the vortex loop of Fig. 19 produces downwash velocities within the enclosed region. These downwash velocities, which are observable in the tuft pattern of Fig. 20, act at the wing itself. They make the wing appear to be flying through air which is itself descending, and this results in what



21. **A vorticity meter whose axis is aligned with the flow shows the streamwise component of vorticity. It is here located behind the trailing edge of a wing, near the wing tip. Flow is from left to right.**



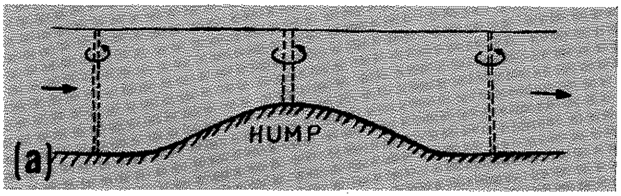
22. **Trailing vortex system from a marine propeller in a water tunnel, made visible by cavitation in the vortex cores. (Courtesy M.I.T. Propeller Tunnel.)**

is called induced drag. The work of the forward-moving wing as it moves against this induced drag force accounts for the kinetic energy being fed into the constantly-lengthening system of trailing vortices.

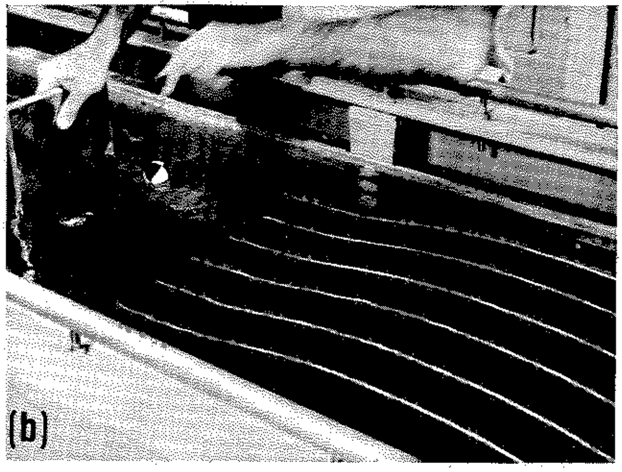
A plausible explanation for the V formation of migrating birds is that each bird takes advantage of the upwash velocities in the trailing vortex systems of the ones forward of it. Each bird behind the leader flies on an ascending induced air current, while the leader has not only his own induced drag, but additional induced drag due to the downwash of all the birds behind him.

The vortex laws of Helmholtz come from the dynamical equations of motion. Therefore anything we deduce from the vortex laws can also be deduced, although perhaps not as conveniently, from the pressure field. With the lifting wing, for example, air can leak around the tip from the high-pressure region below to the low-pressure region above. This leakage produces a transverse flow which accounts for the tip vortex.

To show the effect of stretching vortex lines, water is caused to flow over a hump (Fig. 23), and a vortex with vertical axis is observed. As it approaches the crest of the hump, the spin of the ping-pong ball decreases. This agrees with Helmholtz's third statement,



23a. Water flows sub-critically from left to right over a hump in an open channel. The water depth first decreases to the top of the hump, then increases.



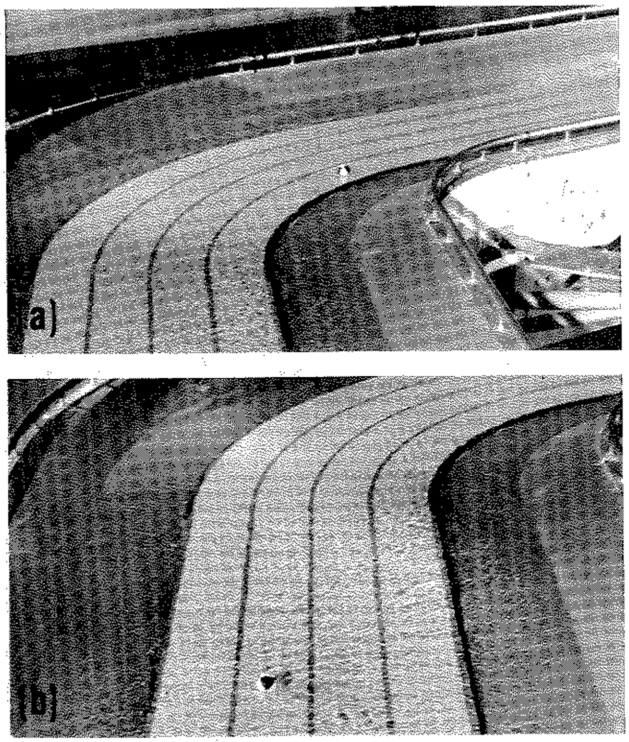
23b. Two vertical vortices are made by a plate with sharp edges, which is then withdrawn, and a ping-pong ball is placed in the dimple of one of the vortices. The rate of spin of the ball gives an indication of the vorticity in the vortex core.

inasmuch as the lengths of the vertical lines of fluid to which the vorticity is attached are also decreasing. When the vortex goes down the hump, the lengths of the vortex lines increase, and the rate of spin of the ball is seen to increase. The change in spin rate as the ball goes down the hump, however, is not as strong as the change going up the hump. That is because viscosity is always acting to *decrease* the rate of spin of the ping-pong ball. There is a simple mechanical explanation of this experiment. When a vertical thread of fluid moves up the hump, its length decreases, but the volume of the thread remains the same; hence its diameter increases. For moment of momentum to be conserved, the spin rate must decrease. When the vertical thread of fluid goes down the hump, it stretches and becomes thinner. Accordingly, the rate of spin increases. A figure skater or ballet dancer knows this mechanical trick instinctively. She speeds up in a pirouette by moving her arms and legs inward to decrease her moment of inertia.

Turbulent flows are full of vorticity. The vortex lines are like tangled spaghetti. The mutually-induced velocities of these vortex lines cause some of them to lengthen, and this lengthening produces a finer-grained turbulence with higher velocity gradients. This makes for added viscous dissipation. As L. F. G. Richardson has put it, "Big whirls make little whirls,/which feed on their velocity./Little whirls make lesser whirls,/and so on, to viscosity."

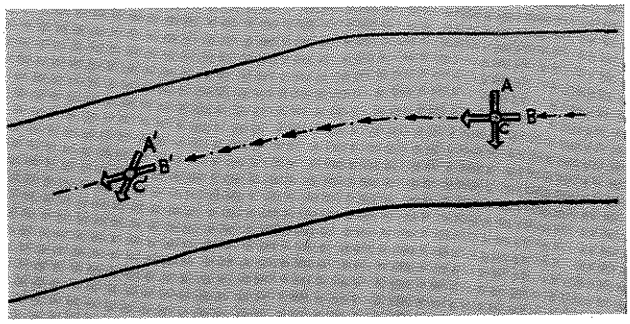
Secondary Flow

The generation of secondary flows is illustrated in the curved channel flow of Fig. 24. Upstream, the flow is parallel to the walls of the channel. Because of



24. Water flows around a bend, from upper right to lower left, in an open channel of trapezoidal cross-section. (a) A floating ping-pong ball entering the bend, near the inside of the bend. (b) The same ball a few seconds later. It is approaching the outer wall.

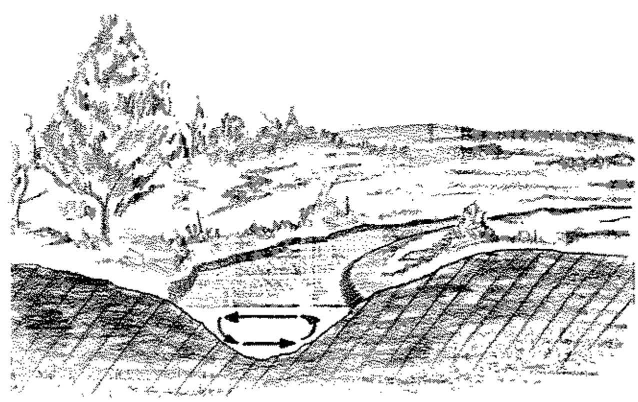
the boundary layer on the bottom of the channel the upstream flow has a horizontal component of vorticity in a direction transverse to the flow, but the vorticity components in the vertical and streamwise directions are both zero. When the streamwise vorticity meter shown in Fig. 21 is aligned with the flow at the exit of the bend, it spins, showing that somehow a streamwise component of vorticity has been generated in the bend. The same result is shown by the drift of the ping-pong ball to the outer side of the bend, while heavy oil droplets rolling along the floor drift to the inside of the bend. To see why, consider three perpendicular fluid lines A, B and C, in the upstream position (Fig. 25). If viscous forces are neglected compared with inertial forces in the bend, the vortex lines may be thought of as frozen to these three fluid lines. Notice, though, that upstream there is no vertical component



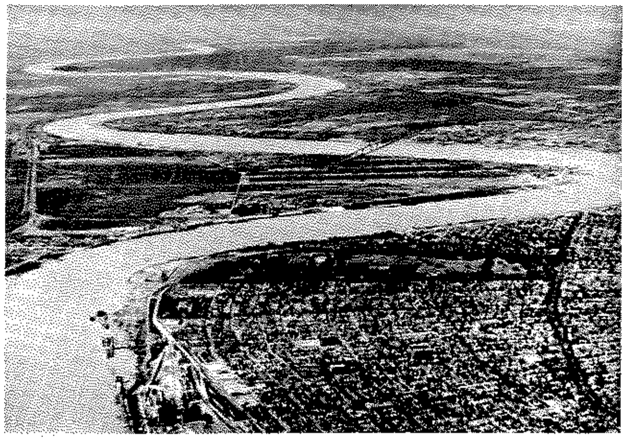
25. Generation of streamwise vorticity in a channel bend.

of vorticity C, nor any streamwise component of vorticity B. There is only a transverse component of vorticity A associated with the viscous boundary layer on the floor of the channel. Now, to first order, C moves along the center streamline to C', so there is no vertical component of vorticity at C'. This means that the *average* turning of the lines A and B must be zero. Line B follows its streamline, rotating counterclockwise, hence line A must rotate clockwise by an equal amount into position A'. One consequence is that the velocity at the inside of the bend is greater than at the outside. But, more interestingly, the vortex line A' now has *two* components: one transverse to the local flow, which existed upstream, and a new component *along* the flow. This streamwise component — the secondary vorticity — swirls the flow clockwise as one looks downstream, and explains the observed motions of the vorticity meter, the ping-pong ball, and the oil droplets.

Such secondary flows often occur in curved channels. When a river goes around a bend (Fig. 26), the secondary flow erodes the outer bank and deposits sand and pebbles on the inner bank. This tends to accentuate the curve of the bend. This may be one of the mechanisms by which exaggerated cases of river meandering occur (Fig. 27). When a pipe is curved, there

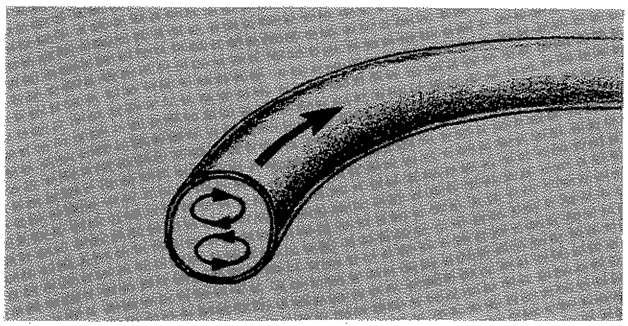


26. Secondary flow currents in a river bend.



27. Meandering of a river.

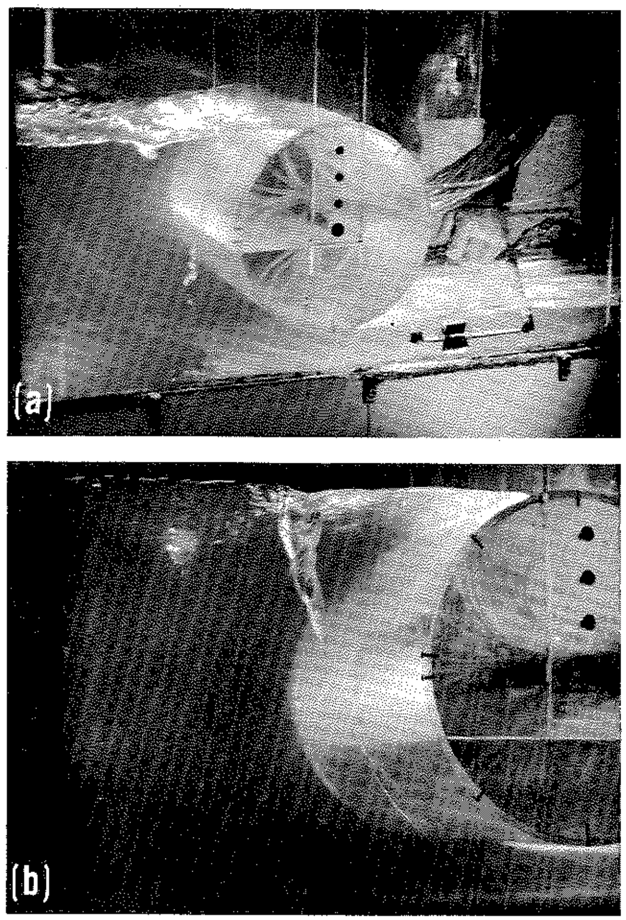
are two cells of secondary flow (Fig. 28). These secondary flows carry high-energy fluid from the middle of the pipe to the walls, thereby increasing the frictional losses.



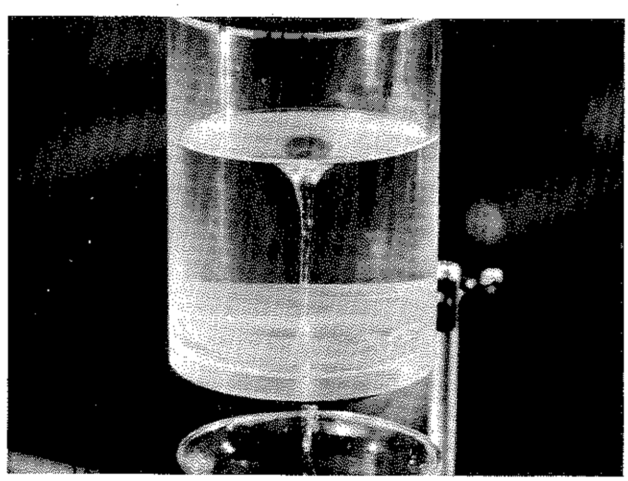
28. Secondary flow in a curved pipe.

The vortex laws are also illustrated by flow in an open channel when it is forced to pass under a transverse circular cylinder (Fig. 29). Strong vortices are developed from the vertical vorticity present in the viscous boundary layers on the two side walls. When a vertical vortex line generated by viscosity on a side wall is convected downstream with the fluid to which it is frozen, it is not only bent around the cylinder but also greatly stretched. The stretching of the vortex line intensifies the vorticity and makes a vertical vortex core visible upstream of the cylinder. The bending of the line produces a streamwise component of vorticity downstream of the cylinder.

Vertical vorticity was introduced by stirring in Fig. 30. Then the plug was pulled from the bottom. A strong vortex formed very quickly. A vertical column of fluid initially on the axis is also a vortex line. When the flow begins, this column is enormously lengthened and the vorticity increases proportionately, according to Helmholtz's third law. A similar mechanism underlies the formation of tornadoes as air with angular momentum flows inwards to the ascending vortex core. The centrifugal pressure field of the vortex creates such



29. Water flowing from left to right in a horizontal channel passes under a transverse circular cylinder which acts as a sluice gate. Vertical vorticity is present in the boundary layers on the vertical side walls, having been generated by viscosity over a considerable distance upstream. The stretching of vortex lines is evidenced by the vertical vortex core upstream of the cylinder (b). The bending of the vortex lines is evidenced by the spinning of the streamwise vorticity meter held in the downstream flow (a).



30. A beaker of water with a small hole in the bottom, initially plugged. The water is stirred with a rod, and the plug is then removed.

a very low pressure at its center that houses over which the eye of the tornado passes literally explode.

In the bathtub vortex experiment of Fig. 13, the fluid initially had a very small vorticity in inertial space due to the turning of the earth. When the plug was pulled the vertical fluid thread on the axis was enormously lengthened and the vorticity in inertial space was strengthened proportionately, finally to the point where we could see it in the reference frame of the earth. Photographs of the earth's cloud cover taken from orbiting satellites show that similar events occur in the earth's atmosphere on a grand scale.

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