MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF PHYSICS
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PROBLEM SET 11
Post date: Thursday, May 7th
Due date: Thursday, May 14th

1. Formation of a black hole: Oppenheimer-Snyder collapse

An objection to the Schwarzschild black hole solution is that it is "eternal" — it exists from $t = -\infty$ to $t = \infty$. One might worry that its interesting features are a consequence of this rather artificial construction. In this problem, we will see how a Schwarzschild black hole forms in the collapse of a simple, non-singular physical object, demonstrating that the exterior spacetime corresponds to something that can form from a reasonable initial matter distribution. Take our initial "star" to be composed of pressureless dust. The star is initially spherical, of initial radius R_*, of mass M, and composed of an isotropic and homogeneous matter distribution.

By Birkhoff's theorem, the exterior is simply described by the Schwarzschild metric:

$$ds^2_{r>R_*} = -\left(1 - \frac{2GM}{r}\right) dt^2 + \frac{dr^2}{1 - 2GM/r} + r^2 d\Omega^2 \ .$$

Since the interior is spatially isotropic and homogeneous, it is perfectly described by the Robertson-Walker metric. Since the star clearly must collapse under its own self gravity, we use the closed RW metric to describe the interior:

$$ds^2_{r<R_*} = -d\tau^2 + a^2(\tau) R_0^2 \left(d\chi^2 + \sin^2\chi d\Omega^2\right) \ .$$

(We use different notation for interior and exterior time: The exterior time t is Schwarzschild time, convenient very far away, but bad in the strong field; the interior time τ denotes the proper time experienced by an element of dust inside the star.)

(a) [6 pts] The evolution of the scale factor for a closed RW line element turns out to have a simple, closed form solution. Show that the parametric solution

$$a = \frac{a_{\max}}{2}(1 + \cos\eta) \ , \qquad \tau = \frac{a_{\max} R_0}{2}(\eta + \sin\eta) \ ,$$

with $0 \le \eta \le \pi$, solves the Friedmann equations for $k = 1$ assuming ρ is given by pressureless dust matter. Since a always appears multiplied by the lengthscale R_0 in the line element, you may set $a_{\max} = 1$. Find a relationship between the initial density ρ_0 and the lengthscale R_0.

(b) [4 pts] This solution for the interior time coordinate τ is only good up to $\tau = \pi R_0/2$. What happens to the interior solution after that?

Now need examine the star's surface from the external viewpoint:

(c) [6 pts] Consider a purely radial "orbit" (i.e., a trajectory with no angular momentum: $L = 0$). For a given energy per unit mass E, find the radius R at which the radial velocity goes to zero.

We will use this solution to define the "orbital energy" of a dust element at the surface of the star as it begins to collapse.

(d) [8 pts] Using the radial geodesic equation for Schwarzschild and the relationship between E and R from (b), write down an integral for the proper time τ it takes for the star's surface to fall from its initial radius R_* to r. You should find

$$\tau = -\int_{R_*}^{r} \frac{dr'}{\sqrt{2GM/r' - 2GM/R_*}} \, .$$

(Minus due to the infalling motion.) By introducing the parameterization

$$r = \frac{R_*}{2}\left(1 + \cos\eta\right) \, , \qquad 0 \leq \eta \leq \pi \, ,$$

show that this integral can be evaluated to give

$$\tau \;=\; \sqrt{\frac{R_*^3}{8GM}}(\eta + \sin\eta) \, .$$

Now match the inner and outer coordinate systems: Require the star's circumference be the same in both systems for all η, and require the two expressions for the proper time τ experienced by a fluid element on the star's surface be the same for all η.

(e) [8 pts] By enforcing these two conditions, determine the lengthscale R_0 and the Robertson-Walker radius of the star χ_*.

For the remainder of the problem, assume the star's initial radius is $R_* = 5GM$.

A black hole's event horizon is a null surface: It is "generated" by null geodesics whose coordinate locations are $r = 2GM$ for all time. The event horizon of a black hole that forms in collapse is "generated" by the null geodesic that begins at the star's center and reaches the surface just as the surface passes through $r = 2GM$; at that point, by Birkhoff's theorem this horizon "generator" will remain at $r = 2GM$ for all time.

(f) [8 pts] Determine the time τ at which the horizon generator leaves the center of the star.

Hint: It is easiest to solve for this geodesic by noting that the parametric solution for the closed RW spacetime allows us to write

$$ds^2 = a(\eta)^2 R_0^2 \left(-d\eta^2 + d\chi^2 + \sin^2\chi \, d\Omega^2\right) \, .$$

An outward propagating null geodesic thus obeys $d\chi/d\eta = 1$. Using the value of χ you found for the star's surface, it should be easy to integrate backwards from the moment the star's surface crosses $r = 2GM$ to determine the value of η at which the horizon generator leaves the star's center. You then just need to convert to τ.

(g) [5 pts] On a spacetime diagram, sketch the evolution of the star's surface and of the event horizon.

Historical note: This calculation was first done by Robert Oppenheimer and Hartland Snyder in 1939; it showed that collapse produces singularities, but that those singularities are "hidden," causally disconnected from all observers by what we now call an event horizon. John Wheeler didn't trust their 1939 calculation; Charles Misner and David Beckedorff first did the calculation outlined in this problem (as a part of Beckedorff's 1962 senior thesis at Princeton), which played a large role in converting Wheeler into a leading apostle of black hole physics. (Many thanks to Misner for educating me about this history, prompted by discovering this problem set via google!) Links to these articles will be provided on the course webpage around May 14th.

2. Consider a static, spherical star cluster in which all stars move in circular orbits. Ignore collisions between stars (i.e., approximate the stars as non-interacting dust). Adopt Schwarzschild-type coordinates with $r = 0$ at the center of the cluster and write the metric in the form

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2 d\Omega^2 \ .$$

(Historical note: A cluster of this type is known as an "Einstein cluster," and was analyzed by Albert Einstein in 1939.)

(a) [6 pts] Find $e^{2\Lambda}$ and $d\Phi/dr$ in terms of $m = \int_0^r 4\pi\rho r^2 dr$, where $\rho = \rho(r)$ is the stars' mass density in the cluster. (We assume that there are enough stars that a continuum treatment is accurate.) Your final equations should be similar to the TOV equations but with one rather crucial difference.

Hint 1: The cluster can be regarded as a "star" in which the restoring force against gravity is orbital motion ("centrifugal force", intuitively) rather than pressure. We assume that there are enough bodies in the cluster that, averaging over their individual motions, the cluster can be regarded as homogeneous, isotropic, and static.

Hint 2: From the circular motion of the bodies that make up the cluster, you should be able to convince yourself that $T_{tt} \neq 0$, but $T_{rr} = 0$. Furthermore, the density ρ is *defined* as $T_{\mu\nu}v^\mu v^\nu$, where $\vec{v}$ is the 4-velocity of a *static* observer. Since we have $v^\mu \doteq (e^{-\Phi}, 0, 0, 0)$, we deduce that $T_{tt} = \rho e^{2\Phi}$. Re-examine the derivation of the equations describing a static fluid star (Carroll Sec. 5.8; note that Carroll uses α for Φ and β for Λ). Modify that derivation and infer $e^{2\Lambda(r)}$ and $d\Phi/dr$.

(b) [6 pts] Define an appropriate effective potential $V_{\rm eff}(r)$. Use it to determine the energy per unit mass $\hat{E}$ and angular momentum per unit mass $\hat{L}$ of a star in the cluster. Your answer should be expressed in terms of r, $m(r)$, and $\Phi(r)$. Determine the orbital frequency $\Omega \equiv d\phi/dt = (d\phi/d\tau)/(dt/d\tau)$.

(c) [6 pts] Use $V_{\rm eff}$ to analyze the stability of orbits of individual stars in the cluster. What local condition must $Gm(r)/r$ satisfy if all orbits at r are to be stable?

(d) [6 pts] Apply the above results to a homogeneous cluster of total mass M and radius R. [Homogeneous means $\rho(r) = \text{const}$, so $m(r) = M(r/R)^3$ for $r \leq R$; you will need to use this to solve for $\Phi(r)$ to complete this part of the problem.] Find the maximum value of GM/R if *all* orbits are to be stable.

(e) [6 pts] Find the cluster with maximal GM/R, compute the redshift of photons emitted from the cluster's surface, and from its center. When quasars were first discovered, their typical redshift was on the order of $z \sim 0.3$. Could a cluster of this type explain this redshift? Today, quasars are measured with redshifts as high as $z \simeq 6.5$. How well does the relativisic cluster hypothesis explain these quasars?

3. Periastron precession

In lecture, we showed that the following equations govern the motion of a test body in the Schwarzschild metric:

$$\begin{aligned}
\left(\frac{dr}{d\tau}\right)^2 &= \hat{E}^2 - V_{\rm eff}(r)\ , \qquad \text{where} \\
V_{\rm eff}(r) &= \left(1 - \frac{2GM}{r}\right)\left(1 + \frac{\hat{L}^2}{r^2}\right)\ ; \\
\frac{d\phi}{d\tau} &= \frac{\hat{L}}{r^2}\ ; \\
\frac{dt}{d\tau} &= \frac{\hat{E}}{1 - 2GM/r}\ .
\end{aligned}$$

We will now manipulate these equations to calculate the precession angle of a relativistic orbit. It is useful to reparameterize the radius: Write

$$r = \frac{p}{1 + e\cos\psi}\ .$$

This reparameterization *defines* the orbit's eccentricity e. In the Newtonian limit, p is the orbit's semi-latus rectum [related to the semi-major axis by $p = a(1 - e^2)$], and ψ is an angle called the *true anomoly*. As ψ goes from 0 to 2π, r oscillates from $r_{\rm min}$ to $r_{\rm max}$ and back, where

$$r_{\rm min} = \frac{p}{1+e}\ , \qquad r_{\rm max} = \frac{p}{1-e}\ .$$

(a) [12 pts] The radii $r_{\rm min}$ and $r_{\rm max}$ are turning points: $\dot{r}$ switches sign, passing through zero. From the rule $\dot{r} = 0$ for $r = r_{\rm min,\ max}$, compute $\hat{E}(p, e)$, $\hat{L}(p, e)$. (It's actually easiest to solve for $\hat{E}^2$ and $\hat{L}$; you should then argue about what sign of the square root to take.) Make sure they reduce to the correct form as $e \to 0$.

(b) [8 pts] Compute

$$\frac{d\phi}{d\psi} = \frac{d\phi/d\tau}{dr/d\tau}\frac{dr}{d\psi}$$

The result should be rather simple, of the form $\sqrt{\text{stuff 1}/(\text{stuff 2} + \text{stuff 3}\cos\psi)}$. If your answer is a mess, something has gone awry.

(c) [5 pts] Expand your answer in powers of $1/p$ and integrate:

$$\begin{aligned}
\Delta\phi &= \phi \text{ accumulated over a full radial orbit} \\
&= \int_0^{2\pi} \frac{d\phi}{d\psi} d\psi\ .
\end{aligned}$$

Your answer, to leading order in $1/p$, should take the form

$$\Delta\phi = 2\pi + \delta\phi\ .$$

Einstein found $\delta\phi = 6\pi GM/a(1-e^2)$, where a is semi-major axis. The numerical value of this angle precisely matched the historical anomoly in Mercury's orbit precession, making him very excited. Hopefully you reproduce his result!