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Assignment #11 Solutions

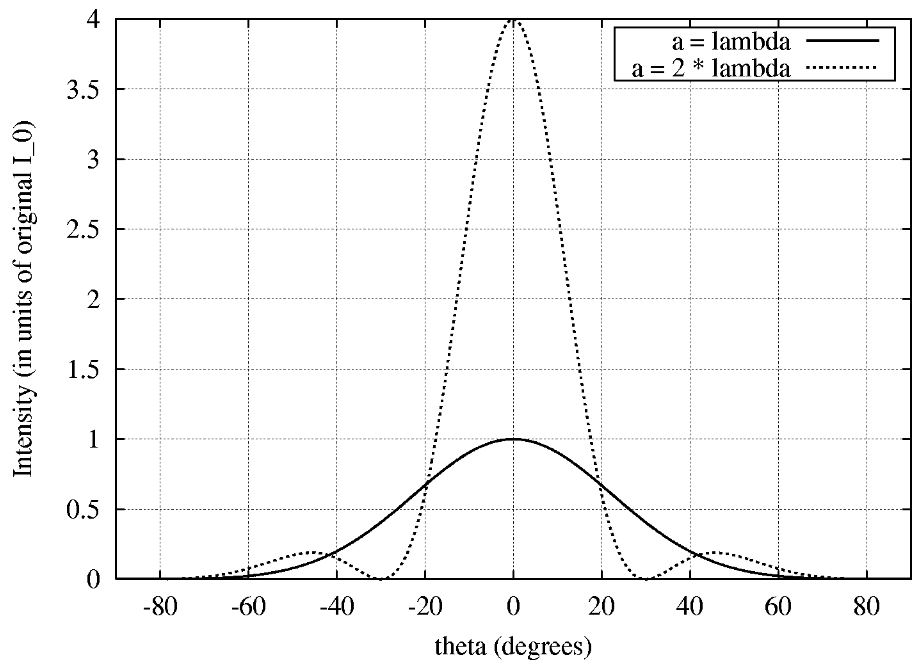
Problem 11.1

Single slit diffraction. (Giancoli 36-9.)

(a) If you double the width of a single slit, you will double the electric field wave amplitude E_0 at the center of the screen. Since light intensity at the center of the screen I_0 is proportional to E_0^2, I_0 will increase by a factor of 4.

(b) Energy conservation is not violated because the intensity becomes more sharply peaked around $\theta = 0$. For example, notice that the angular position of the first minimum (given by $\sin\theta = \lambda/a$) will be reduced when the slit width a is doubled (keeping wavelength λ fixed).

The following plot of I_θ vs. θ illustrates the situation, with $a = \lambda \rightarrow a = 2\lambda$. For each curve, the total power transmitted is proportional to the area under the curve. One can see that although the $a = 2\lambda$ curve has a peak four times higher than the $a = \lambda$ curve, the area underneath is more like twice the area under the $a = \lambda$ curve, in agreement with the expectation of energy conservation.



Problem 11.2

Gratings – Physics and candle light – Home experiment II.

(a) The slit separation is the inverse of the number of lines per millimeter:

$$d = \frac{1}{1000\,\text{mm}^{-1}} = 10^{-6}\,\text{m} = 10{,}000\,\text{Å} \quad .$$

(b) The angular positions of the maxima from a diffraction grating are given by Giancoli Equation (36-13) (p. 900):

$$\sin\theta = \frac{m\lambda}{d} \quad .$$

The first-order ($m = 1$) positions for red light and blue light are

$$\theta_{1,\text{red}} = \arcsin\left(\frac{6{,}300}{10{,}000}\right) = 39.1^\circ \ , \quad \theta_{1,\text{blue}} = \arcsin\left(\frac{4{,}500}{10{,}000}\right) = 26.7^\circ \quad ,$$

and the angle between first-order red and blue lines is

$$\Delta\theta_{1,\text{red}-\text{blue}} = \theta_{1,\text{red}} - \theta_{1,\text{blue}} = 12.3^\circ \quad .$$

(c) $\sin\theta$ cannot exceed 1, so for any given wavelength λ and line spacing d, the highest possible order m is the greatest integer such that $m\lambda/d$ is still less than or equal to 1. $m\lambda/d \leq 1 \Rightarrow m \leq d/\lambda$, thus for red and blue light we find

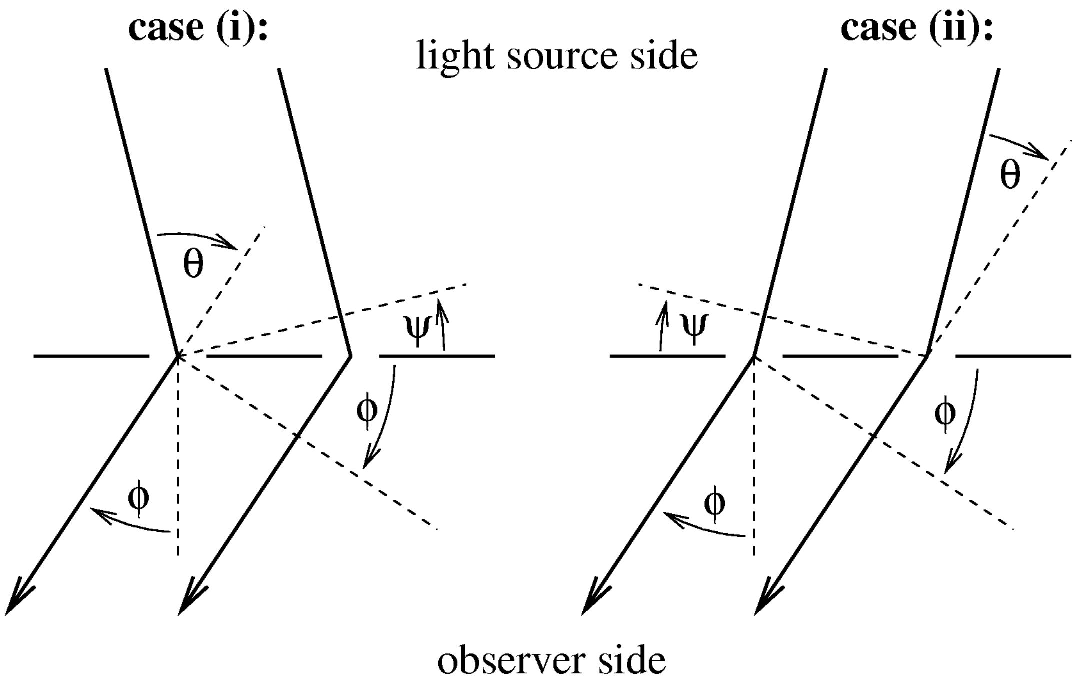
$$\begin{aligned} d/\lambda_{\text{red}} &= \frac{10{,}000}{6{,}300} \simeq 1.6 \quad \Longrightarrow \quad m_{\text{max, red}} = 1 \\ d/\lambda_{\text{blue}} &= \frac{10{,}000}{4{,}500} \simeq 2.2 \quad \Longrightarrow \quad m_{\text{max, blue}} = 2 \quad . \end{aligned}$$

(d) The zero-order spectrum is white; it contains all colors, unseparated in angle.

(e) Observing candlelight in a dark room, I can see the first-order maxima in red and blue quite easily. I can also make out the second-order blue maximum (although it is much fainter). This agrees with the prediction of (c).

(f) For $\ell = 24$ inches, I find $L = 34\frac{1}{4}$ inches, giving $\tan\theta = \ell/L = 0.70 \ \Rightarrow \ \theta = 35^\circ$ for the position of the first-order red maximum: not *exactly* 39°, but fairly close. What I've judged as "red" is probably more "orange" than 6,300-Å light.

(g) Here we must consider two apparently distinct cases. Suppose we are observing the higher-order maxima that appear to the right of the light source. In case (i), we rotate the grating by an angle ψ about the vertical by moving the left side of the grating towards the source and the right side away from the source. In case (ii) we move the left side away from the source and the right side towards the source. We take θ to be the angle between the light source (i.e. the zero-order maximum) and the higher-order maxima: this is the angle that we observe most directly. Let us also define ϕ to be the angle of the diffracted rays with respect to the grating. Let d be the slit spacing. The situation is shown in the following diagram:



First consider case (i). With respect to the ray on the left, the ray on the right has an additional pathlength $d \sin \psi$ on the source side of the grating and another additional pathlength $d \sin \phi$ on the observer side. The condition for constructive interference (and thus the appearance of a maximum) is therefore $d \sin \phi + d \sin \psi = m\lambda$. Next consider case (ii). Now the *left-hand* ray has an additional pathlength $d \sin \psi$ on the source side, while the right-hand ray still has an additional pathlength $d \sin \phi$ on the observer side. The condition for the appearance of a maximum is now $d \sin \phi - d \sin \psi = m\lambda$.

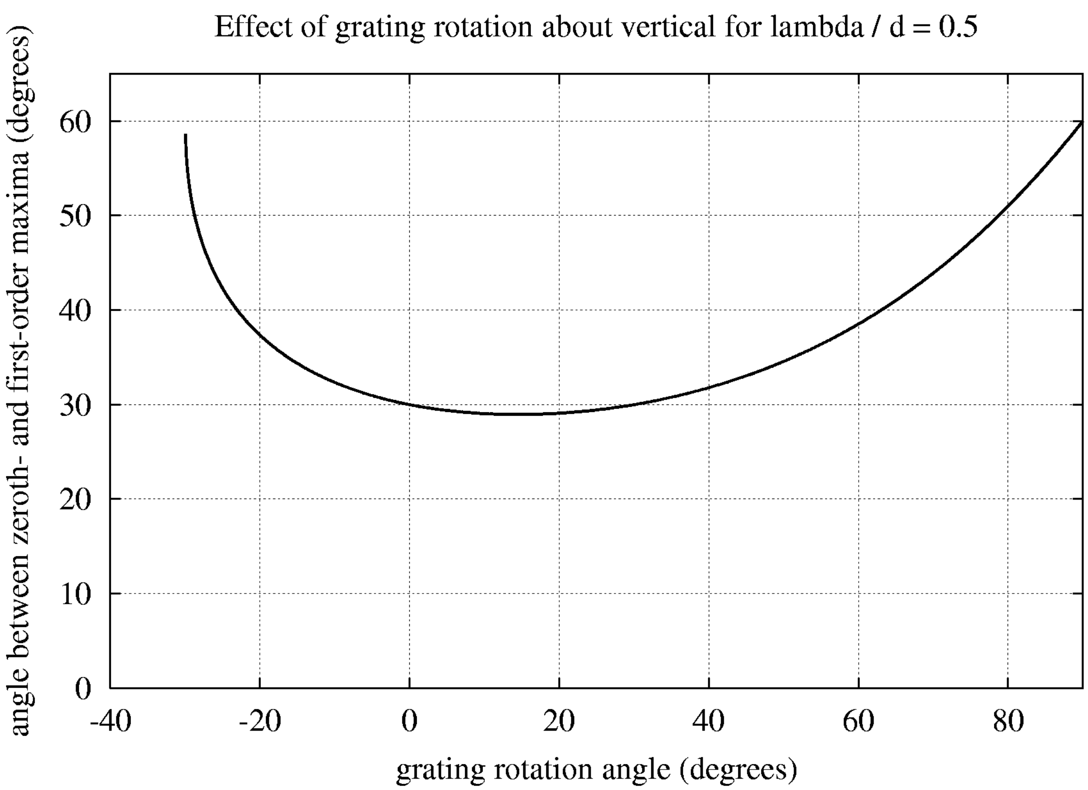
From the geometry, we deduce that $\theta = \phi + \psi$ in case (i) and $\theta = \phi - \psi$ in case (ii). The constructive interference criteria for cases (i) and (ii) can then be re-expressed as

$$\begin{aligned} \sin(\theta - \psi) + \sin \psi &= \frac{m\lambda}{d} \quad \text{for case (i)}, \\ \sin(\theta + \psi) - \sin \psi &= \frac{m\lambda}{d} \quad \text{for case (ii)}. \end{aligned}$$

If we (somewhat arbitrarily) take case (i) to correspond to positive ψ values and case (ii) to correspond to negative ψ values, both conditions take the form of the case (i) condition, which solves to

$$\theta = \arcsin\left(\frac{m\lambda}{d} - \sin\psi\right) + \psi \quad .$$

Let's now take some numerical values for our grating: $m = 1$, $\lambda = 5{,}000$ Å, and $d = 10{,}000$ Å, giving $m\lambda/d = 0.5$. If we plot θ as a function of ψ (shown below), we see that θ in fact slightly *decreases* at first when we increase ψ from zero, then begins increasing again. θ increases immediately when we go towards negative ψ values from $\psi = 0$. Also note that there is a singularity at $\psi = -30°$, beyond which the argument of the inverse-sine function exceeds 1 and the first-order maximum is no longer present.



Problem 11.3

Diffraction, interference, and angular resolution of 2-element interferometers.

(a) Giancoli Equation (36-10) (p. 897) gives the angular resolution of a single radio telescope of diameter $100\,\text{ft} = 3048\,\text{cm}$ at a wavelength of $21\,\text{cm}$:

$$\theta = \frac{1.22\lambda}{D} = \frac{(1.22)21}{3048} = 8.4 \times 10^{-3}\,\text{rad} = 0.48^\circ \simeq 1700\,\text{arcsec} \quad .$$

($1^\circ = 3600\,\text{arcsec}$.)

(b) To determine the angular resolution of the pair of telescopes operating as an interferometer, we make an analogy to the double-slit experiment. The angular resolution will be set by the position of the first minimum in the diffraction/interference pattern described by Giancoli Equation (36-9) (p. 894). Since the effective "slit separation" of $d = 1\,\text{km}$ is much greater than the effective "slit width" of 100 ft, this first minimum will occur when the $\cos^2(\delta/2)$ factor first goes to zero:

$$\frac{\pi}{2} = \frac{\delta}{2} = \frac{\pi}{\lambda} d \sin\theta \quad \Longrightarrow \quad \sin\theta = \frac{\lambda}{2d} = \frac{0.21}{(2)1000} = 1.05 \times 10^{-4} \quad .$$

$\sin\theta \cong \theta$ for such small angles, thus we have

$$\theta = 1.05 \times 10^{-4}\,\text{rad} = (6.0 \times 10^{-3})^\circ = 22\,\text{arcsec}$$

for the angular resolution—a big improvement.

Problem 11.4

Destructive interference of sound. (Giancoli 36-55.)

We can idealize this as a single-slit diffraction problem with sound waves instead of light waves. Our slit width is $a = 0.88\,\text{m}$ and the wave frequency is $f = 750\,\text{Hz}$. We'll take $v = 344\,\text{m/s}$ for the speed of sound in air. The whistle will not be heard clearly at the angular position(s) of minima in the diffraction pattern (Giancoli Equation (36-2), p. 889):

$$\sin\theta = m\frac{\lambda}{a} = m\frac{v}{fa} = m\frac{(344)}{(0.88)(750)} = (0.52)m \quad .$$

The only non-zero integer m that yields a solution is $m = 1$ (for $\sin\theta \leq 1$ necessarily). So in an ideal situation, the whistle would go unheard at

$$\theta = \arcsin(0.52) = 31^\circ \quad .$$

Problem 11.5

Resolving power of the human eye. (Giancoli 36-65.)

The diffraction-limit angular resolution of the human eye (assumed diameter 5.0 mm) at a wavelength of 500 nm is

$$\theta = \frac{1.22\lambda}{D} = \frac{(1.22)(5\times10^{-7})}{(5.0\times10^{-3})} = 1.22\times10^{-4}\,\text{rad} = 0.42\,\text{arcmin} \qquad (1^\circ = 60\,\text{arcmin}).$$

(a) Considering only diffraction effects, the human eye will be able just barely to distinguish two car headlights when they are at a distance L such that their separation d subtends an angle θ equal to the above angular resolution limit:

$$\text{(small angle:)} \quad d = L\theta \quad \Longrightarrow \quad L = \frac{d}{\theta} = \frac{2.0\,\text{m}}{1.22\times10^{-4}} \simeq 16\,\text{km} \quad .$$

(b) Again considering only diffraction effects, the minimum angular star separation that the human eye could discern would be the above 0.42-arcminute value. The eye's *actual* angular resolution is more like 1 arcminute due to the finite density of photoreceptors in the retina. In fact, as demonstrated in lectures, most students were not even able to resolve two lights of equal strength which were 1 arc-minute apart.

Problem 11.6

Resolving power of optical telescopes.

(a)

$$\theta = \frac{1.22\lambda}{D} = \frac{(1.22)(4.5\times10^{-7})}{2.4} = 2.3\times10^{-7}\,\text{rad} = 0.047\,\text{arcsec} \quad .$$

(b) As discussed in lectures and also noted in Giancoli Example 36-5 (p. 897), the angular resolution of ground-based telescopes is limited to about $\frac{1}{2}$ arcsecond (at best) by turbulence in the Earth's atmosphere.

(c) As mentioned in lectures (see also Giancoli Example 36-5), the angular resolution of the Hubble Space Telescope is limited not by atmospheric effects but by diffraction. Therefore, for a wavelength of 4.5×10^{-7} m, it has the 0.047-arcsecond resolution of part (a).

(d) (See preceding parts.)

Problem 11.7

Doppler shift of light I. (Giancoli 37-56.)

The Doppler-shift formula for receding sources (Giancoli Equation (37-15b), p. 943) can be solved algebraically for the relative velocity of recession:

$$f = f_0\sqrt{\frac{c-v}{c+v}} \quad \Longrightarrow \quad v = \left[\frac{1-(f/f_0)^2}{1+(f/f_0)^2}\right] c \quad .$$

We are given that $f_0 - f = 0.797 f_0 \;\Rightarrow\; f/f_0 = 0.203$. So then,

$$v = \left[\frac{1-(0.203)^2}{1+(0.203)^2}\right] c = 0.921c = 2.76 \times 10^8\,\mathrm{m/s} \quad .$$

Problem 11.8

Doppler shift of light II. (Giancoli 37-59.)

If $v \ll c$, then $v/c \ll 1$. Working to first-order accuracy in v/c and making use of Giancoli Equation (37-15a) (p. 943):

$$\begin{aligned}
\frac{\Delta\lambda}{\lambda} &= \frac{\lambda - \lambda_0}{\lambda} \\
&= 1 - \frac{\lambda_0}{\lambda} \\
&= 1 - \sqrt{\frac{c-v}{c+v}} \\
&= 1 - \sqrt{\frac{1-v/c}{1+v/c}} \\
&\simeq 1 - \sqrt{(1-v/c)(1-v/c)} \\
&\simeq 1 - \sqrt{1-2v/c} \\
&\simeq 1 - (1-v/c) \\
&= \frac{v}{c} \quad .
\end{aligned}$$

END