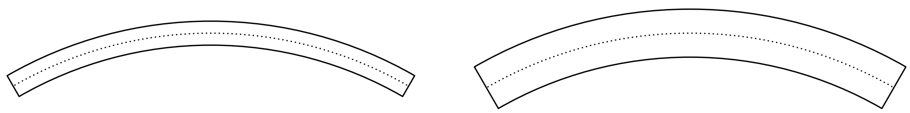

Above the neutral line the springs are extended. Below the neutral line, the springs are compressed. The amount of extension is proportional to the distance from the neutral line. Each spring in the thin block corresponds to a spring in the thick block that is twice as far away from the neutral line. The spring in the thick block has twice the extension (or compression) of its partner in the thin block. So the spring in the thick block stores four times the energy of its partner spring in the thin block. Furthermore, the thick block has twice as many layers as does the thin black, so each spring in the thin block has two partners, with identical extension, in the thick block. So the thick block stores eight times the energy of the thin block, for the same deflection y.

Thus

$$\frac{k_{\mathrm{thick}}}{k_{\mathrm{thin}}} = 8.$$

This factor of 8 results from multiplying the thickness by 2. In general,

$$k \propto h^3.$$

Since $\omega \sim \sqrt{k/m}$, and

$$\frac{m_{\mathrm{thick}}}{m_{\mathrm{thin}}} = 2,$$

the frequency ratio is

$$\frac{\omega_{\mathrm{thick}}}{\omega_{\mathrm{thin}}} = \sqrt{\frac{8}{2}} = 2.$$

In general, $m \propto h$ so

$$\frac{\omega_{\mathrm{thick}}}{\omega_{\mathrm{thin}}} = \sqrt{\frac{h^3}{h}} = h.$$

Frequency is proportional to thickness!

Let's check this analysis by looking at its consequences and comparing with experimental data from a home experiment.

10.1.2 Xylophone

My daughter got a toy xylophone from her uncle. Its slats have these dimensions:

	ℓ
C	12.2 cm
D	11.5
E	10.9
F	10.6
G	10.0
A	9.4
B	8.9
C′	8.6

Our analysis of how frequency depends on thickness can explain this pattern of how frequency depends on length. The method is to use dimensional analysis with proportional reasoning (scaling).

Rather than finding the frequency direction, I analyze the stiffness. The mass is easy, so split that part off of the calculation of the frequency. The block's spring constant k depends on its material properties – here, the Young's modulus Y – and on its dimensions. So the variables are k, Y, and length l, width w, and thickness (height) h.

How many independent dimensions are contained in those variables? How many independent dimensionless groups can be formed from those variables?

These five variables are composed of two independent dimensions. These dimensions could be length and force: Stiffness is force per length, and Young's modulus is force per area. Five variables based on two independent dimensions form three independent dimensionless groups. The goal is to find k, so I include k in only one group. That group contains Y to divide out the dimensions of mass. Since Young's modulus is force per area, and stiffness is force per length, the ratio k/Yh is dimensionless. The three lengths for the size of the block easily make two more dimensionless groups: for example, h/l and w/l. Then

$$\frac{k}{Yh} = f\left(\frac{h}{l}, \frac{w}{l}\right).$$

Guess the function f (except for a dimensionless constant).

What we know about stiffness versus thickness, along with proportional reasoning, is enough to solve for f, except for a dimensionless constant.

Proportional reasoning helps determine the dependence on the dimensionless group w/l. Imagine doubling the width w. Equivalently, I glue together two identical blocks along the long, thin edge. When the new block is bent, the individual blocks contains equal energy, so the new block contains twice the energy of an original block. Therefore, doubling the width doubles the stiffness; in symbols, $k \propto w$. In the general form

$$\frac{k}{Yh} = f\left(\frac{h}{l}, \frac{w}{l}\right).$$

w appears only in the group w/l, and k appears on the right in the first power. So the general form simplifies to

$$\frac{k}{Yh} = \frac{w}{l} \cdot f\left(\frac{h}{l}\right).$$

To guess the new function f, I use what I know about stiffness versus thickness, that $k \propto h^3$. Therefore the left side, k/Yh, is proportional to h^2. On the right side the only source of h is from f, which can play with h but only via the ratio h/l. So

$$f\left(\frac{h}{l}\right) \sim \left(\frac{h}{l}\right)^2.$$

Combining these deductions gives

$$\frac{k}{Yh} \sim \frac{w}{l}\left(\frac{h}{l}\right)^2 = \frac{wh^2}{l^3}$$

and

$$k \sim Yw\left(\frac{h}{l}\right)^3.$$

The stiffness and mass determine the frequency. The mass is $m = \rho w l h$. So

$$\omega \sim \sqrt{\frac{k}{m}} \sim \sqrt{\frac{Y}{\rho}}\,\frac{h}{l^2}.$$

As a quick check, this result is consistent with the earlier calculation that frequency is proportional to thickness. And it contains a new result: $\omega \propto l^{-2}$.

Problem 10.1 Effect of width
Is it physically plausible that the width w does not affect the frequency ω?

▶ *Is this data consistent with the prediction that $\omega \propto l^{-2}$?*

Before doing an extensive analysis, I check the easy case of the octave. The lower and higher C notes are a factor of 2 apart in frequency. If the scaling prediction is correct, the respective slat lengths should be a factor of $\sqrt{2}$ apart. The length ratio is $12.2/8.6 \sim 1.419$, which is very close to $\sqrt{2}$. The general pattern is that fl^2 should be invariant. To check, here is the same table with frequencies, which are computed by assuming that the A above middle C is at 440 Hz (concert A), and with a column for fl^2:

	ℓ	f
C	12.2	261.6
D	11.5	293.6
E	10.9	329.6
F	10.6	349.2
G	10.0	392.0
A	9.4	440.0
B	8.9	493.8
C′	8.6	523.2

The proposed invariant is, experimentally, almost constant.

10.2 Waves

Ocean covers most of the earth, and waves roam most of the ocean. Waves also cross puddles and ponds. What makes them move, and what determines their speed? By applying and extending the techniques of approximation, we analyze waves. For concreteness, this section refers mostly to water waves but the results apply to any fluid. The themes of section are: *Springs are everywhere* and *Consider limiting cases.*

10.2.1 Dispersion relations

The most organized way to study waves is to use **dispersion relations**. A dispersion relation states what values of frequency and wavelength a wave can have. Instead of the wavelength λ, dispersion relations usually connect frequency ω, and wavenumber k, which is defined as $2\pi/\lambda$. This preference has an basis in order-of-magnitude reasoning. Wavelength is the the distance the wave travels in a full period, which is 2π radians of oscillation. Although 2π is dimensionless, it is not the ideal dimensionless number, which is unity. In 1 radian of oscillation, the wave travels a distance