Chapter 10
Springs

Every physical process contains a spring! The main example in this chapter is waves, which illustrate springs, discretization, and special cases – a fitting, unified way to end the book.

10.1 Musical tones

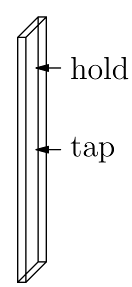
10.1.1 Wood blocks

Here is a home musical experiment that illustrates proportional reasoning and springs. First construct, or ask a carpenter or a local lumber yard to construct, two wood blocks made from the same larger wood plank. Mine have these dimensions:

1. $30\,\text{cm} \times 5\,\text{cm} \times 1\,\text{cm}$; and
2. $30\,\text{cm} \times 5\,\text{cm} \times 2\,\text{cm}$.

The blocks are identical except in their thickness: $2\,\text{cm}$ vs $1\,\text{cm}$.

Now tap the thin block at the center while holding it lightly toward the edge, and listen to the musical note. If you do the same experiment to the thick block, will the pitch (fundamental frequency) be higher than, the same as, or lower than the pitch when you tapped the thin block?

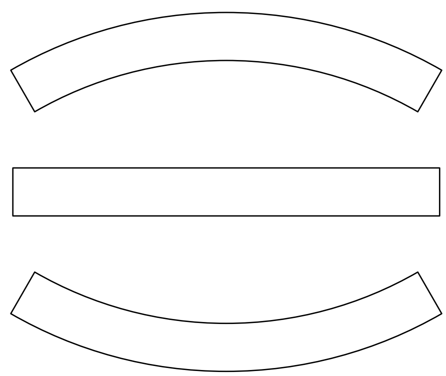


You can answer this question in many ways. The first is to do the experiment. It would be nice either to predict the result before doing the experiment or to explain and understand the result after doing the experiment.

One argument is that the block is a resonant object, and the wavelength of the sound depends on the thickness of the block. In that picture, the thick block should have the longer wavelength and therefore the lower frequency. A counterargument, based on a different model of how the sound is made, is that the thick block is stiffer, so it vibrates faster. On the other hand, the thick block is more massive, so it vibrates more slowly. Perhaps these two effects – greater stiffness but greater mass – cancel each other, leaving the frequency unchanged?

I'll do the experiment right now and tell you the result. The thick block has a higher pitch. So the resonant-cavity model is probably wrong. Instead, the stiffness probably more than overcomes the mass.

A spring model explains this result and even predicts the frequency ratio. In the spring model, a wood block is made of wood atoms connected by chemical bonds, which are springs. As the block vibrates, it takes shapes like these (in a side view):



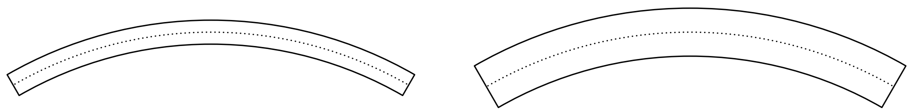
The block is made of springs, and it acts like a big spring. The middle position is the equilibrium position, when the block has zero potential energy and maximum kinetic energy. The potential energy is stored in stretching and compressing the bonds. Imagine deforming the block into a shape like the top shape. Since the block is a big spring, to produce the vertical deflection y requires an energy $E \sim ky^2$, where k is the stiffness of the block.

To find how k depends on the thickness h, deflect the thin and thick blocks by the same amount y, and compare the stored energies:

$$\frac{k_{thick}}{k_{thin}} = \frac{E_{thick}}{E_{thin}},$$

because y is, by construction, the same for the thick and thin blocks.

Here are the blocks, with the dotted line showing the neutral line, which is the line without compression or extension:



Above the neutral line the springs are extended. Below the neutral line, the springs are compressed. The amount of extension is proportional to the distance from the neutral line. Each spring in the thin block corresponds to a spring in the thick block that is twice as far away from the neutral line. The spring in the thick block has twice the extension (or compression) of its partner in the thin block. So the spring in the thick block stores four times the energy of its partner spring in the thin block. Furthermore, the thick block has twice as many layers as does the thin black, so each spring in the thin block has two partners, with identical extension, in the thick block. So the thick block stores eight times the energy of the thin block, for the same deflection y.

Thus

$$\frac{k_{\text{thick}}}{k_{\text{thin}}} = 8.$$

This factor of 8 results from multiplying the thickness by 2. In general,

$$k \propto h^3.$$

Since $\omega \sim \sqrt{k/m}$, and

$$\frac{m_{\text{thick}}}{m_{\text{thin}}} = 2,$$

the frequency ratio is

$$\frac{\omega_{\text{thick}}}{\omega_{\text{thin}}} = \sqrt{\frac{8}{2}} = 2.$$

In general, $m \propto h$ so

$$\frac{\omega_{\text{thick}}}{\omega_{\text{thin}}} = \sqrt{\frac{h^3}{h}} = h.$$

Frequency is proportional to thickness!

Let's check this analysis by looking at its consequences and comparing with experimental data from a home experiment.

10.1.2 Xylophone

My daughter got a toy xylophone from her uncle. Its slats have these dimensions: