Chapter 4

Symmetry and conservation

When symmetry can be applied to a problem, it often greatly simplifies the problem – and at no cost in accuracy. A classic example is a story about the young Carl Friedrich Gauss. The story is perhaps an urban legend, but it is so instructive that it ought to be true.

When Gauss was 3 years old, the story goes, his schoolteacher wanted to occupy the young students for a good while. So he asked them to compute

$$S = 1 + 2 + 3 + \cdots + 100.$$

To the teacher's surprise, Gauss returned in just a few minutes claiming that the sum is 5050. Was he right? If so, how did he do it so quickly?

Gauss noticed that the sum remains fixed if the terms are added backwards, from last to first. In other words,

$$S' = 100 + 99 + 98 + \cdots + 1$$

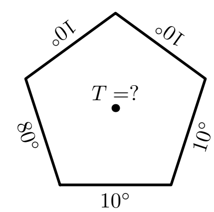
equals S. Then add these two ways to compute S:

$$\begin{array}{r@{\;}l} S = & 1 + 2 + 3 + \cdots + 100 \\ +S = & 100 + 99 + 98 + \cdots + 1 \\ \hline 2S = & 101 + 101 + \cdots + 101. \end{array}$$

In this form, $2S$ is easy to compute because it is 100 copies of 101. So $2S = 100 \times 101$ and $S = 50 \times 101 = 5050$.

Gauss found a symmetry, and it tremendously simplified the problem. In order to extract a general pattern to reuse in other areas, let's try symmetry in diverse examples.

4.1 Heat flow



Imagine a metal sheet, perhaps aluminum foil, cut in the shape of a regular pentagon. Attach heat sources and sinks to the edge that hold the five edges at the temperatures marked on the figure. After enough time passes, the temperature distribution in the pentagon stops changing ('comes to equilibrium'). What then is the temperature at the center of the pentagon?

A brute-force analytic solution is difficult. Heat flow is described by the following second-order partial differential equation:

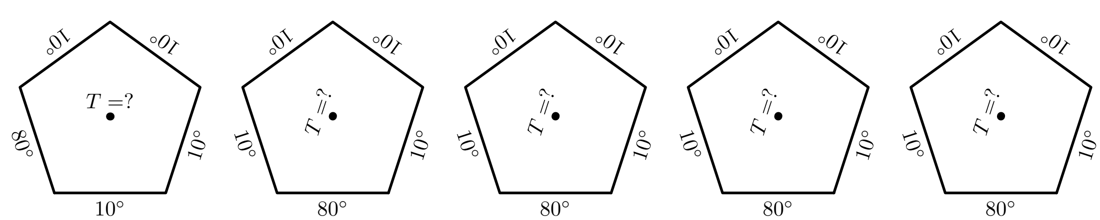
$$\kappa \nabla^2 T = \frac{\partial T}{\partial t},$$

where T is the temperature as a function of position and time, and κ is a constant known as the thermal diffusivity. Waiting makes time derivatives approach zero (everything eventually settles down), so in our problem the right side is zero. Therefore, the equation simplifies to

$$\kappa \nabla^2 T = 0.$$

Alas, even this simpler time-independent equation has simple solutions only for a few simple boundaries. A pentagon, even a regular pentagon, is not among those boundaries.

Symmetry, however, makes the solution flow. Rotating the pentagon about its center does not change the temperature at the center. Nature, in the person of the heat equation, does not care in what direction our coordinate system points. Mathematically stated, the laplacian operator ∇^2 is rotation invariant. So these five orientations of the pentagon behave identically:



Now stack these sheets (mentally), adding the temperatures that lie on top of each other to make the temperature profile of a new metal supersheet. On this new sheet, each edge has temperature

$$T_{edge} = 80° + 10° + 10° + 10° + 10° = 120°.$$

To solve this resulting temperature distribution, there is no need to solve the heat equation. Since all the edges are held at 120°, the temperature throughout the sheet is 120°.

That information is enough to solve the original problem. The symmetry operation is a rotation about the center of the pentagon, so the centers overlap when the plates are stacked atop one another. Since the stacked plate has a temperature of 120° throughout, and the centers of the five stacked sheets align, each center is at $T = 120°/5 = 24°$.

To find transferable ideas, compare the symmetry solutions to Gauss's sum and to the pentagon temperature. Both problems looked complex at first glance. Gauss's sum had many terms in it, all different. The pentagon problem seemed to require solving a difficult differential equation. Both problems contained a symmetry operation. In Gauss's sum, the symmetry operation flipping the sum around. In the pentagon problem, the symmetry operation rotated the pentagon by 72°. In both problems, the symmetry operation left an important quantity unchanged: the sum S or the temperature T_{center}. And this invariance became the key to solving the problem simply.

A moral of these two examples is: When there is change, look for what does not change. In other words, look for invariants. Alternatively, if those quantities are given (e.g. the sum S or temperature at the center), look for operations that leave them unchanged. In other words, look for symmetries.

4.2 Cube solitaire

Here is a game of solitaire that illustrates the theme of this chapter. The following cube starts in the configuration in the margin; the goal is to make all vertices be multiples of three simultaneously. The moves are all of the same form: Pick any edge and increment its two vertices by one. For example, if I pick the bottom edge of the front face, then the bottom edge of the back face, the configuration becomes the first one in this series, then the second one: