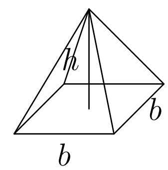
However, as a physicist I am embarrassed by the reasoning. This example teaches me a valuable lesson about theorems and proofs: judge the proof not just the theorem. Even if you disagree with the conclusion, remember the general lesson that a correct conclusion does not validate a dubious argument.

6.2 Pyramid volume

The last example showed the value of dimensions in economics. The next example shows that dimensions are also useful in mathematics. What is the volume of this square-based pyramid? Here are several choices:



1. $\frac{1}{3}bh$
2. $b^3 + h^2$
3. b^4/h
4. bh^2

Let's take the choices in turn. The first choice, $bh/3$, has dimensions of area rather than volume. So it cannot be right. The second choice, $b^3 + h^2$, begins with a volume in the b^3 term but falls apart with the h^2, which has dimensions of area. Since it adds an area to a volume – the crime of dimension mixing – it cannot be right. The third choice, b^4/h, has dimensions of volume, so it might be correct. It even increases as b increases, which is a good sign. However, the volume should increase as h increases – a proportional-reasoning argument – whereas this choice indicates that the volume decreases as h increases! So it cannot be right.

The final choice, bh^2, has correct dimensions and increases as h or b increases. Does it increase by the right amounts? Imagine drilling into the pyramid from the top and dividing it into thin cores or volume elements. If the height of the pyramid doubles, then each vertical volume element doubles in volume; so the volume of the pyramid should double. In symbols, $V \propto h$. But bh^2 quadruples when h doubles, so that choice cannot be right.

The requirement that $V \propto h$ together with the requirement that V have dimensions of length cubed means that the missing item in $V \propto h$ is an area. The only way to make an area from b is to make b^2 perhaps times a dimensionless constant. So

$$V \sim hb^2.$$

The missing dimensionless constant is hidden in the twiddle ~ sign. Alternatively, the ratio V/hb^2 is dimensionless.

This method of deducing the volume requires remembering hardly any arbitrary data. It requires these ingredients:

1. Using vertical volume elements to find out that $V \propto h$.
2. Using dimensions along with $V \propto h$ to show that $V \sim hb^2$.
3. Remembering the correct dimensionless constant.

The first two steps are logic and do not require arbitrary data. Instead they use reasoning methods that you use elsewhere (so there's no marginal cost to remember them). The third step requires seemingly arbitrary data. However, in **Chapter 7** on special cases, I'll show you how to determine the constant elegantly without even needing an integral.

Then the volume requires no memory. Arbitrary data is, by definition, impossible to compress. Dimensions, and more generally our techniques for handling complexity, are a form of data compression or entropy reduction [21]. One way to look at learning is as data compression. So dimensions, and our other techniques, enhance learning.

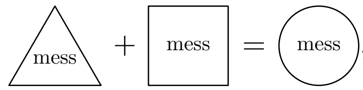
There's an old saying: Tell the truth; there's less to remember. The similar moral here is: *Use dimensions (and proportional reasoning); there's less to remember!*

6.3 Dimensionless groups

Dimensionless ratios are useful. For example, in the oil example, the ratio of the two quantities has dimensions; in that case, the dimensions of the ratio are time (or one over time). If the authors of the article had used a dimensionless ratio, they might have made a valid comparison.

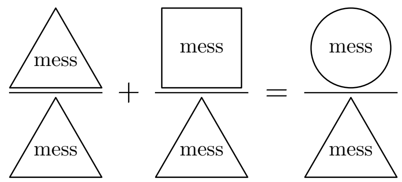
This section explains why dimensionless ratios are the only quantities that you need to think about; in other words, that there is no need to think about quantities with dimensions.

To see why, take a concrete example: computing the energy E to produce lift as a function of distance traveled s, plane speed v, air density ρ, wingspan L, plane mass m, and strength of gravity g. Any true statement about these variables looks like



where the various messes mean 'a horrible combination of E, s, ν, ρ, L, and m.

As horrible as that true statement is, it permits the following rewriting: Divide each term by the first one (the triangle). Then

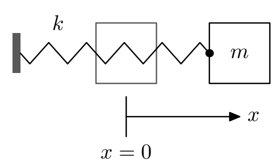


The first ratio is 1, which has no dimensions. Without knowing the individual messes, we don't know the second ratio; but it has no dimensions because it is being added to the first ratio. Similarly, the third ratio, which is on the right side, also has no dimensions.

So the rewritten expression is dimensionless. Nothing in the rewriting depended on the particular form of the true statement, except that each term has the same dimensions.

Therefore, *any true statement can be rewritten in dimensionless form.*

Dimensionless forms are made from dimensionless ratios, so all you need are dimensionless ratios, and you can do all your thinking with them. Here is a familiar example to show how this change simplifies your thinking. This example uses familiar physics so that you can concentrate on the new idea of dimensionless ratios.



The problem is to find the period of an oscillating spring–mass system given an initial displacement x_0, then allowed to oscillate freely. **Section 5.2** gave a proportional-reasoning analysis of this system. The relevant variables that determine the period T are mass m, spring constant k, and amplitude x_0. Those three variables completely describe the system, so any true statement about period needs only those variables.

Var	Dim	What
T	T	period
m	M	mass
k	MT^{-2}	spring constant
x_0	L	amplitude

Since any true statement can be written in dimensionless form, the next step is to find all dimensionless forms that can be constructed from T, m, k, and x_0. A table of dimensions is helpful. The only tricky entry is the dimensions of a spring constant. Since the force from the spring is $F = kx$, where x is the displacement, the dimensions of a spring constant are the dimensions of force divided by the dimensions of x. It is convenient to have a notation for the concept of 'the dimensions of'. In that notation,

$$[k] = \frac{[F]}{[x]},$$

where [quantity] means the dimensions of the quantity. Since $[F] = MLT^{-2}$ and $[x] = L$,

$$[k] = MT^{-2},$$

which is the entry in the table.

These quantities combine into many – infinitely many – dimensionless combinations or groups:

$$\frac{kT^2}{m}, \frac{m}{kT^2}, \left(\frac{kT^2}{m}\right)^{25}, \pi\frac{m}{kT^2}, , \ldots.$$

The groups are redundant. You can construct them from only one group. In fancy terms, all the dimensionless groups are formed from one *independent* dimensionless group. What combination to use for that one group is up to you, but you need only one group. I like kT^2/m.

So any true statement about the period can be written just using kT^2/m. That requirement limits the possible statements to

$$\frac{kT^2}{m} = C,$$

where C is a dimensionless constant. This form has two important consequences:

1. The amplitude x_0 does not affect the period. This independence is also known as simple harmonic motion. The analysis in **Section 5.2** gave an approximate argument for why the period should be independent of the amplitude. So that approximate argument turns out to be an exact argument.

2. The constant C is independent of k and m. So I can measure it for one spring–mass system and know it for all spring–mass systems, no matter the mass or spring constant. The constant is a universal constant.

The requirement that dimensions be valid has simplified the analysis of the spring–mass system. Without using dimensions, the problem would be to find (or measure) the three-variable function f that connects m, k, and x_0 to the period:

$$T = f(m, k, x_0).$$

Whereas using dimensions reveals that the problem is simpler: to find the function h such that

$$\frac{kT^2}{m} = h().$$

Here $h()$ means a function of no variables. Why no variables? Because the right side contains all the other quantities on which kT^2/m could depend. However, dimensional analysis says that the variables appear only through the combination kT^2/m, which is already on the left side. So no variables remain to be put on the right side; hence h is a function of zero variables. The only function of zero variables is a constant, so $kT^2/m = C$.

This pattern illustrates a famous quote from the statistician and physicist Harold Jeffreys [25, p. 82]:

> A good table of functions of one variable may require a page; that of a function of two variables a volume; that of a function of three variables a bookcase; and that of a function of four variables a library.

Use dimensions; avoid tables as big as a library!

Dimensionless groups are a kind of invariant: They are unchanged even when the system of units is changed. Like any invariant, a dimensionless group is an abstraction (**Chapter 3**). So, looking for dimensionless groups is recipe for developing new abstractions.

6.4 Hydrogen atom

Hydrogen is the simplest atom, and studying hydrogen is the simplest way to understand the **atomic theory**. Feynman has explained the importance of the atomic theory in his famous lectures on physics [, p. 1-2]:

> If, in some cataclysm, all of scientific knowledge were to be destroyed, and only one sentence passed on to the next generations