This simple code – simple to understand and simple to write – expands into 34 lines of tedious, error-prone MetaPost boxes code (not shown here to avoid boring you). The moral is: Let a computer, which rarely makes errors, do the translation, and do your thinking using the higher-level abstractions.

3.4 Example: Operators

The next abstraction is two levels more abstract than ordinary numbers. Ordinary numbers are

Operators turn functions into functions. The space of functions is itself vast and complex, so operators are complex beasts. Ignoring most of that complexity makes operators act like ordinary numbers. Although this abstraction is leaky, it leaks so rarely that we can figure out a lot by adopting it and charging ahead fearlessly. 'Be approximately right rather than exactly wrong' (attributed to John Tukey and John Maynard Keynes).

3.4.1 Derivative operator

A familiar operator is the derivative. Here is a differential equation for the motion of a damped spring, in a suitable system of units:

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + x = 0,$$

where x is dimensionless position, and t is dimensionless time. Imagine x as the amplitude divided by the initial amplitude; and t as the time multiplied by the frequency (so it is radians of oscillation). The dx/dt term represents the friction, and its plus sign indicates that friction dissipates the system's energy. A useful shorthand for the d/dt is the operator D. It is an operator because it operates on an object – here a function – and returns another object. Using D, the spring's equation becomes

$$D^2x(t) + 3Dx(t) + x(t) = 0.$$

The tricky step is replacing d^2x/dt^2 by D^2x, as follows:

$$D^2x = D(Dx) = D\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}.$$

The analogy comes in solving the equation. Pretend that D is a number, and do to it what you would do with numbers. For example, factor the equation. First, factor out the $x(t)$ or x, then factor the polynomial in D:

$$(D^2 + 3D + 1)x = (D + 2)(D + 1)x = 0.$$

This equation is satisfied if either $(D + 1)x = 0$ or $(D + 2)x = 0$. The first equation written in normal form, becomes

$$(D + 1)x = \frac{dx}{dt} + x = 0,$$

or $x = e^{-t}$ (give or take a constant). The second equation becomes

$$(D + 2)x = \frac{dx}{dt} + 2x = 0,$$

or $x = e^{-2t}$. So the equation has two solutions: $x = e^{-t}$ or e^{-2t}.

The example above introduced D and its square, D^2, the second derivative. You can do more with the operator D. You can cube it, take its logarithm, its reciprocal, and even its exponential. Let's look at the exponential e^D. It has a power series:

$$e^D = 1 + D + \frac{1}{2}D^2 + \frac{1}{6}D^3 + \cdots.$$

That's a new operator. Let's see what it does by letting it operating on a few functions. For example, apply it to $x = t$:

$$(1 + D + D^2/2 + \cdots)t = t + 1 + 0 = t + 1.$$

And to $x = t^2$:

$$(1 + D + D^2/2 + D^3/6 + \cdots)t^2 = t^2 + 2t + 1 + 0 = (t + 1)^2.$$

And to $x = t^3$:

$$(1 + D + D^2/2 + D^3/6 + D^4/24 + \cdots)t^3 = t^3 + 3t^2 + 3t + 1 + 0 = (t + 1)^3.$$

It seems like, from these simple functions (extreme cases again), that $e^D x(t) = x(t + 1)$. You can show that for any power $x = t^n$, that

$$e^D t^n = (t + 1)^n.$$

Since any function can, pretty much, be written as a power series, and e^D is a linear operator, it acts the same on any function, not just on the powers. So e^D is the successor function: It replaces $x(t)$ by $x(t + 1)$.

3.4.2 Successor operator

Now that we know how to represent the successor operator in terms of derivatives, let's give it a name: S, and use it. It is useful in finding sums and evaluating derivatives. Let's first use it for evaluating derivatives. Suppose you sample a function and want to compute its derivative at one of the points.

$$D = \ln(S) = \ln(1 + (S - 1)) = (S - 1) + (S - 1)^2/2 + \cdots$$

3.4.3 Euler–MacLaurin Summation

Suppose you have a function $f(n)$ and you want to find the sum $\sum f(k)$. Never mind the limits for now. It's a new function of n, so summation, like integration, takes a function and produces another function. It is an operator, $\sum$. Let's figure out how to represent it in terms of familiar operators. To keep it all straight, let's get the limits right. Let's define it this way:

$$F(n) = (\sum f)(n) = \sum_{-\infty}^{n} f(k).$$

So $f(n)$ goes into the maw of the summation operator and comes out as $F(n)$. Look at $SF(n)$. On the one hand, it is $F(n + 1)$, since that's what S does. On the other hand, S is, by analogy, just a number, so let's swap it inside the definition of $F(n)$:

$$SF(n) = (\sum Sf)(n) = \sum_{-\infty}^{n} f(k + 1).$$

The sum on the right is $F(n) + f(n + 1)$, so

$$SF(n) - F(n) = f(n + 1).$$

Now factor the $F(n)$ out, and replace it by $\sum f$:

$$((S - 1) \sum f)(n) = f(n + 1).$$

So $(S - 1) \sum = S$, which is an implicit equation for the operator $\sum$ in terms of S. Now let's solve it:

$$\sum = \frac{S}{S - 1} = \frac{1}{1 - S^{-1}}.$$

Since $S = e^D$, this becomes

$$\sum = \frac{1}{1 - e^{-D}}.$$

Again, remember that for our purposes D is just a number, so find the power series of the function on the right:

$$\sum = D^{-1} + \frac{1}{2} + \frac{1}{12}D - \frac{1}{720}D^3 + \cdots.$$

The coefficients do not have an obvious pattern. But they are the Bernoulli numbers. Anyway, let's look at the terms one by one to see what the mean. First is D^{-1}, which is the inverse of D. Since D is the derivative operator, its inverse is the integral operator. So the first approximation to the sum is the integral – what we know from first-year calculus.

The first correction is $1/2$. Huh? Are we supposed to add $1/2$ to the integral, no matter what function we are summing? That cannot be right. And it isn't. The $1/2$ is one piece of an operator, and the whole sum is applied to a function. Let's take it in slow motion:

$$\sum f(n) = \int^n f(k)\,dk + \frac{1}{2}f(n) + \cdots.$$

So the first correction is one-half of the final term $f(n)$.

Problem 3.1 Pictorial explanation
Find a pictorial explanation for the $f(n)/2$ term in $\sum f(n)$.

3.4.4 Euler sum

Let's improve the estimate for the Euler sum $\sum_1^\infty n^{-2}$. The first term is 1, the result of integrating. The second term is $1/2$, the result of $f(1)/2$. The third term is $1/6$, the result of $D/12$ applied to n^{-2}. So:

$$\sum_1^\infty n^{-2} \approx 1 + \frac{1}{2} + \frac{1}{6} = 1.666\ldots$$

The true value is $1.644\ldots$, so we're close. The fourth term gives a correction of $-1/30$. So the new value is $1.633\ldots$. The approximation gets better and better!

Let's see where the $\pi^2/6$ comes from, by using analogy at a key step. Look at the function $\sin x$. That intersects the x-axis at $\pm n\pi x$, where $n = 0, 1, 2, \ldots$.

Let's get rid of the π by looking at the function $\sin \pi x$, which has roots at $\pm nx$. Now, $\sin z$ is an **entire function**: It has no infinities – no **poles** – for any z, even for complex z. Polynomials also have no poles. An entire function is analogous to a polynomial: It is an infinite-degree polynomial. Others are e^z and $\sinh z$. Why is the analogy useful? Because your knowledge from the source system helps you generate ideas to use in the destination system. Polynomials are characterized by their zeros, so maybe entire functions are as well. For polynomials, that characterization is done by factoring them. So let's factor entire functions too.

How does $\sin \pi z$ factor? We already have a good idea.

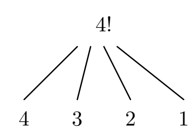
As we'll see in a later chapter, rational functions generalize to what are called **meromorphic functions** in complex analysis: functions with zeros and poles.

3.5 Example: Recursion

Sometimes you make a minilanguage to solve just one problem. The minilanguage or abstraction is reusable, and is reused multiple times in solving that problem. Recursion is an example of this use of abstraction.

A classic example of recursion is computing $n!$. Here is a non-recursive definition of factorial:

$$n! \equiv n \times (n-1) \times (n-2) \times \cdots \times 1.$$



The tree illustrates how to compute 4! using this definition: You multiply 4, 3, 2, and 1.

Then I have a great insight: You notice that $3 \times 2 \times 1$ is also 3!, which is $3 \times 2!$, and so on. This realization turns the flat, seemingly unstructured tree into a tree with a pattern.