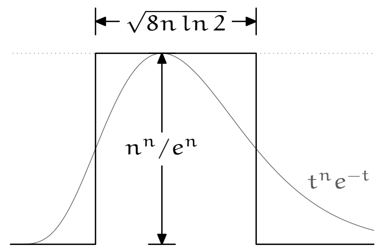
$$e^{f(t)} = \left(\frac{n}{e}\right)^n e^{-(t-n)^2/2n}.$$



The first factor is a constant, the peak height. The second factor is the familiar Gaussian. This one is centered at $t = n$ and contains $1/2n$ in the exponent but otherwise it's just a Gaussian. It falls by a factor of 2 when $(t-n)^2/2n = \ln 2$, which is when

$$t_\pm = n \pm \sqrt{2n \ln 2}.$$

The FWHM is $t_+ - t_-$, which is $\sqrt{8n \ln 2}$. The approximate area under $e^{f(t)}$, which is $n!$, is then

$$n! \approx \left(\frac{n}{e}\right)^n \times \sqrt{8n \ln 2}.$$

This approximation reproduces the most important factors of Stirling's approximation: the n^n in the numerator and the e^n in the denominator. Stirling's approximation contains $\sqrt{2\pi}$ instead of $\sqrt{8 \ln 2}$ – a change of only 6%.

Problem 4.9 Coincidence?

The FWHM approximation for the area under a Gaussian (Section 4.3) was also accurate to 6%. Coincidence?

Problem 4.10 More accurate constant factor

Where does the more accurate constant factor of $\sqrt{2\pi}$ come from?

4.5 Pendulum period

Is it coincidence that g, in units of meters per second squared, is 9.81, very close to $\pi^2 \approx 9.87$? Their proximity suggests a connection. Indeed, they are connected through the original definition of the meter. It was proposed by the the Dutch scientist and engineer Christian Huygens (science and engineering were not separated in the 17th century) – called 'the most ingenious watchmaker of all time' by the great physicist Arnold Sommerfeld

[21, p. 79]. Huygens's portable definition of the meter required only a pendulum clock: Adjust the bob's length l until the pendulum requires $1\,\mathrm{s}$ to swing from one side to the other; in other words, until its period is $T = 2\,\mathrm{s}$. A pendulum's period (for small amplitudes) is $T = 2\pi\sqrt{l/g}$, as shown below, so

$$g = \frac{4\pi^2 l}{T^2}.$$

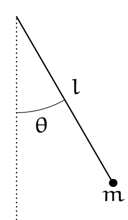
Using the $T = 2\,\mathrm{s}$ standard for the meter,

$$g = \frac{4\pi^2 \mathrm{x} 1\,\mathrm{m}}{4\,\mathrm{s}^2} = \pi^2\,\mathrm{m\,s^{-2}}.$$

So, if Huygens's standard were used today, then g would be π^2 by definition. Instead, it is close to that value. The story behind the difference is rich in physics, mechanical and materials engineering, mathematics, and history; see [22, 23, 24] for several views of a vast and fascinating subject.

Problem 4.11 How is the time measured?
Huygens's standard for the meter requires a way to measure time, and no quartz clocks were available. How could one, in the 17th century, ensure that the pendulum's period is indeed $2\,\mathrm{s}$?

Here our subject is to find how the period of a pendulum depends on its amplitude. The analysis uses all our techniques so far – dimensions (Chapter 2), easy cases (Chapter 3), and discretization (this chapter) – to learn as much as possible without solving differential equations.



Here is the differential equation for the motion of an ideal pendulum (one with no friction, a massless string, and a miniscule bob):

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0,$$

where θ is the angle with respect to the vertical, g is the gravitational acceleration, and l is the mass of the bob.

Instead of deriving this equation from physical principles (see [25] for a derivation), take it as a given but check that it makes sense.

Are its dimensions correct?

It has only two terms, and they must have identical dimensions. For the first term, $d^2\theta/dt^2$, the dimensions are the dimensions of θ divided by T^2 from the dt^2. (With apologies for the double usage, this T refers to the time dimension rather than to the period.) Since angles are dimensionless (see Problem 4.12),

$$\left[\frac{d^2\theta}{dt^2}\right] = T^{-2}.$$

For the second term, the dimensions are

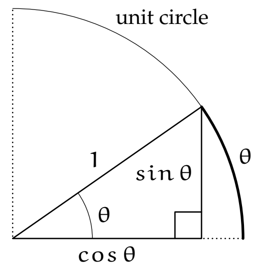
$$\left[\frac{g}{l}\sin\theta\right] = \left[\frac{g}{l}\right] \times [\sin\theta] \, .$$

Since $\sin\theta$ is dimensionless, the dimensions are just those of g/l, which are T^{-2}. So the two terms have identical dimensions.

Problem 4.12 Angles
Why are angles dimensionless?

Problem 4.13 Where did the mass go?
Use dimensions to show that the differential equation cannot contain the mass of the bob (except as a common factor that divides out).

Because of the nonlinear factor $\sin\theta$, solving this differential equation is difficult. One can compute a power-series solution, and call the resulting infinite series a new function. That procedure, when applied to another differential equation, is the origin of the Bessel functions. However, the so-called elementary functions – those built from $\sin$, $\cos$, $\exp$, $\ln$, and powers – do not contain a solution to the pendulum equation.



So, use easy cases to simplify the source of the problem, namely the $\sin\theta$ factor. One easy case is the extreme case $\theta \to 0$. To approximate $\sin\theta$ in that limit, mark θ and $\sin\theta$ on a quarter-section of the unit circle. By definition, θ is the length of the arc. Also by definition, $\sin\theta$ is the altitude of the enclosed right triangle. When θ is small, the arc is almost exactly the altitude. Therefore, for small θ:

$$\sin\theta \approx \theta.$$

It is a tremendously useful approximation.

Problem 4.14 Slightly better approximation
The preceding approximation replaced the arc with a straight, vertical line. A more accurate approximation replaces the arc with the chord (a straight but non-vertical line). What is the resulting approximation for $\sin\theta$, including the θ^3 term?

In this small-θ extreme, the pendulum equation turns into

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0.$$

It looks like the ideal-spring differential equation analyzed in Section 2.5:

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0,$$

where m is the mass and k is the spring constant (the stiffness). Comparing the two equations produces this correspondence:

$$x \to \theta;$$
$$\frac{k}{m} \to \frac{g}{l}.$$

Since the oscillation period for the ideal spring is

$$T = 2\pi\sqrt{\frac{m}{k}},$$

the oscillation period for the pendulum, in the $\theta \to 0$ limit, is

$$T = 2\pi\sqrt{\frac{l}{g}}.$$

▶ *Does this period have correct dimensions?*

Pause to sanity check this result by asking: 'Is each portion of the formula reasonable, or does it come out of left field.' [For non-American readers, left field is one of the distant reaches of a baseball field. To come out of left fields means an idea comes almost out of nowhere, surprising all with its craziness.] The first sanity check is dimensions. They are correct in the approximate spring differential equation; but let's also check the dimensions of the period $T = 2\pi\sqrt{l/g}$ that results from solving the equation. In the symbolic factor $\sqrt{l/g}$, the lengths cancel and leave only T^2 inside the square root. So $\sqrt{l/g}$ is a time – as it should be.

▶ *What about easy cases?*

Another sanity check is easy cases. For example, imagine a huge gravitational field strength g. Then gravity easily and rapidly swings the bob to and fro, making the period tiny. So g should live in the denominator of T – and it does.

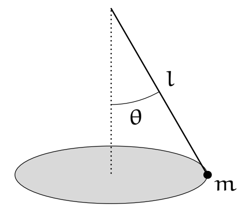
Problem 4.15 Another easy case?
Can you use easy cases to explain why l belongs in the numerator?

▶ *Didn't the 2π come from solving differential equations, contrary to the earlier promise to avoid solving differential equations?*

The dimensions and easy-cases tests confirm the $\sqrt{l/g}$ factor. But how to explain the remaining piece: the numerical factor of 2π that arose from the solution to the ideal-spring differential equation. However, we want to avoid solving differential equations. Can our techniques derive the 2π?

4.5.1 Small amplitudes and Huygens' method

Dimensions and easy cases rarely explain a dimensionless constant. Therefore explaining the factor of 2π probably requires a new idea. It too is due to Huygens. His idea [21, p. 79ff] is to analyze the motion of a conical pendulum: a pendulum moving in a horizontal circle. Although its motion is two dimensional, it is at constant speed, so it is easy to analyze without solving differential equations.



Even if the analysis of the conical pendulum is simple, how is it relevant to the motion of a one-dimensional pendulum?

Projecting the two-dimensional motion onto a screen produces one-dimensional pendulum motion, so the period of the two-dimensional motion is the same as the period of the one-dimensional motion! This statement is slightly false when θ_0 is large. But when θ_0 is small, which is the extreme analyzed here, the equivalence is exact.

To project onto one-dimensional motion with amplitude θ_0, give the conical pendulum the constant angle $\theta = \theta_0$. The plan is to use the angle to find the speed of the bob, then use the speed to find its period.

What is the speed of the bob in terms of l and θ_0?

To find the speed, find the inward force in two ways:

1. To move in a circle of radius r at speed v, the bob requires an inward force

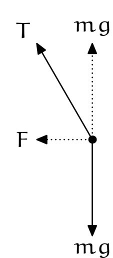
$$F = \frac{mv^2}{r},$$

where m is the mass of the bob (it anyway divides out later).

2. The two forces on the bob are from gravity and from the string tension. Since the bob has zero vertical acceleration – it has no vertical motion at all – the vertical component of the tension force cancels gravity:

$$T\cos\theta_0 = mg.$$

Therefore, the horizontal component of tension is the net force on the mass, so that net force is



$$F = T\sin\theta_0 = \underbrace{T\cos\theta_0}_{mg}\tan\theta_0 = mg\tan\theta_0.$$

Equating these two equivalent expressions for the inward force F gives $mg\tan\theta_0 = mv^2/r$ or $v = \sqrt{gr\tan\theta_0}$. Since the radius of the circle is $r = l\sin\theta_0$, the bob's speed is

$$v = \sqrt{gl\tan\theta_0\sin\theta_0}.$$

Problem 4.16 Check dimensions
Check that $v = \sqrt{gl\tan\theta_0\sin\theta_0}$ has correct dimensions.

The period is the circumference divided by speed:

$$T = \frac{2\pi r}{v} = \frac{2\pi l\sin\theta_0}{\sqrt{gl\tan\theta_0\sin\theta_0}} = 2\pi\sqrt{\frac{l\cos\theta_0}{g}}.$$

As long as θ_0 is small, $\cos\theta_0$ is approximately 1, so $T \approx 2\pi\sqrt{l/g}$. This equation contains a negative result: the absence of θ_0; therefore, period is independent of amplitude (for small amplitudes). This equation also contains a positive result: the magic factor of 2π, courtesy of Huygens and without solving differential equations.

4.5.2 Large amplitudes

The preceding results are valid when the amplitude θ_0 is infinitesimally small. When this restriction is removed, how does the period behave?

▶ *Does the period increase, decrease, or remain constant as θ_0 is increased?*

First reformulate this question in dimensionless form by constructing dimensionless groups (Section 3.4.1). The period T belongs to a dimensionless group $T/\sqrt{l/g}$. Since the amplitude θ_0 is no longer restricted to be near zero, it can have an important effect on period, so θ_0 should also join a dimensionless group. Since angles are dimensionless, θ_0 can make a dimensionless group by itself. With these choices, the problem contains two dimensionless groups (Problem 4.17): $T/\sqrt{l/g}$ and θ_0.

Problem 4.17 Dimensionless groups using the pendulum variables
Check that the period T, length l, gravitational strength g, and amplitude θ_0 produce two independent dimensionless groups.

In constructing two useful groups, why should the period T appear in only one group? For the same purpose, why should θ_0 not appear in the same group as T?

Two dimensionless groups produce this general dimensionless form:

$$\text{one group} = f(\text{other group}),$$

or

$$\frac{T}{\sqrt{l/g}} = f(\theta_0),$$

where f is a dimensionless function. Since $T/\sqrt{l/g}$ goes to 2π as θ_0 (the ideal-spring limit), simplify slightly by pulling out the factor of 2π:

$$\frac{T}{\sqrt{l/g}} = 2\pi h(\theta_0),$$

where the dimensionless function h has the simple endpoint value $h(0) = 1$. The function h contains all the information about how the period of a pendulum depends on its amplitude. In terms of h, the preceding question about the period becomes this question:

▶ *Is the function $h(\theta_0)$ monotonic increasing, monotonic decreasing, or constant?*

This type of question suggests considering easy cases of θ_0: If the question can be answered for any case, the answer identifies a likely trend for the whole amplitude range. Two easy cases are the extremes of the amplitude range. One extreme is already analyzed case $\theta_0 = 0$; it reproduces the differential equation and behavior of an ideal spring. But that analysis does not predict the behavior of the pendulum when θ_0 is nonzero but still small. Since the low-amplitude extreme is not easy to analyze, try the large-amplitude extreme.

▶ *How does the period behave at large amplitudes? What is a large amplitude?*

A large amplitude could be $\theta_0 = \pi/2$. That case is, however, hard to analyze. The exact value of $h(\pi/2)$ is the following awful expression, as can be shown using conservation of energy (Problem 4.18):

$$h(\pi/2) = \frac{\sqrt{2}}{\pi}\int_0^{\pi/2} \frac{d\theta}{\sqrt{\cos\theta}}.$$

Is this expression less than, equal to, or more than 1?! Who knows. The integral looks unlikely to have a closed form, and numerical evaluation is difficult without a computer (Problem 4.19).

Problem 4.18 General expression for h

Use conservation of energy to show that the period of a pendulum with amplitude θ_0 is

$$T(\theta_0) = 2\sqrt{2}\sqrt{\frac{l}{g}}\int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}.$$

In terms of h, the equivalent statement is that

$$h(\theta_0) = \frac{\sqrt{2}}{\pi}\int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}.$$

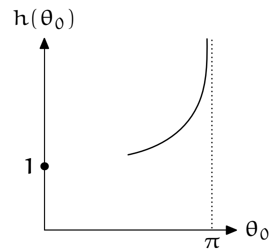
For horizontal release, $\theta_0 = \pi/2$, whereupon

$$h(\pi/2) = \frac{\sqrt{2}}{\pi}\int_0^{\pi/2} \frac{d\theta}{\sqrt{\cos\theta}}.$$

Problem 4.19 Numerical evaluation for horizontal release

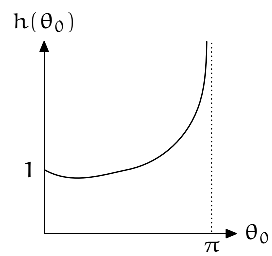
Why do the discretization recipes, such as the ones in Section 4.2 and Section 4.3, fail for the integrals in Problem 4.18?

Use or write a program to evaluate $h(\pi/2)$ numerically.



Since $\pi/2$ was not a helpful extreme, be even more extreme:[3] Try $\theta_0 = \pi$: releasing the pendulum bob from the highest possible point. That release location fails if the pendulum bob is connected to the support point by only a string – the pendulum would collapse downwards rather than oscillate. This behavior is not described by the pendulum differential equation, which assumes that the pendulum bob is constrained to move in a circle of radius l. Fortunately, the experiment is easy to improve, because it is a thought experiment. So, replace the string with a material that can maintain the constraint: Let's splurge on a rigid but massless steel rod. The improved pendulum does not collapse even when $\theta_0 = \pi$.

Balanced at $\theta_0 = \pi$, the pendulum bob will hang upside down forever; in other words, $T(\pi) = \infty$. For smaller amplitudes, the period is finite, so the period most probably increases as amplitude increases toward π. Stated in dimensionless form, $h(\theta_0)$ most probably increases monotonically toward infinity.

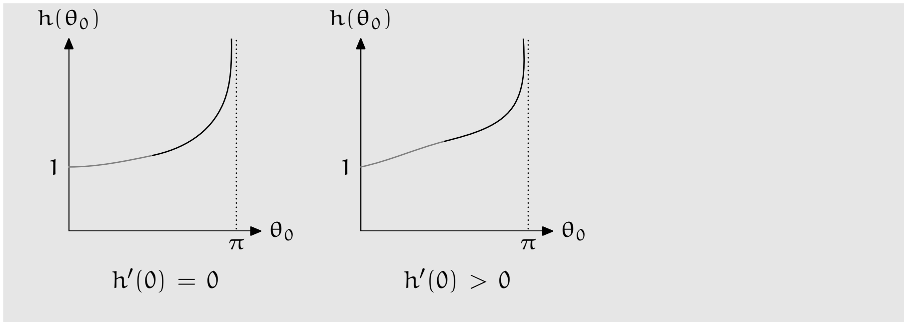


Although monotonic behavior is the simplest assumption, alternative assumptions are possible. For example, for small θ_0, the dimensionless function $h(\theta_0)$ could decrease from 1; then flatten; then increase toward infinity as θ_0 approaches π. Altough possible, such behavior would be surprising compared to the original, pendulum differential equation. What would such a nice, smooth differential equation like the pendulum equation be doing producing such a badly behaved, non-monotonic solution? This complicated behavior is therefore unlikely. As a rule of thumb, assume until proven otherwise that nature does not play nasty tricks.

Problem 4.20 Small but nonzero amplitude

At $\theta_0 = 0$, does $h(\theta_0)$ have zero or positive slope? In other words, which figure is the more likely to be correct:

3 One definition of insanity is repeating an action but expecting a different result.

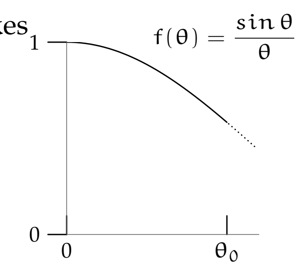


As has been said in arms-control negotiations: 'Trust but verify.' So, while trusting the preceding rule of thumb, verify it by more accurately analyzing the period at small amplitudes.

This analysis seems like it requires solving the original pendulum differential equation,

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0.$$

To avoid this difficult task, let's isolate, encapsulate, and try to mitigate the equation's complexity.



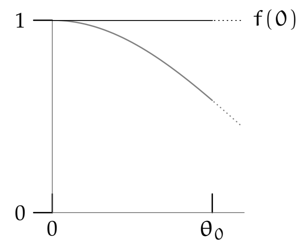
The complexity arises because the $\sin\theta$ factor makes the equation nonlinear. If only that factor were θ, then the equation would be linear and tractable. And $\sin\theta$ is almost θ: The functions θ and $\sin\theta$ match as θ goes to 0. However, as θ grows – i.e. for larger amplitudes – θ and $\sin\theta$ part company. To explicate the comparison, rewrite the differential equation in this form:

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta f(\theta) = 0,$$

where the ratio $f(\theta) \equiv (\sin\theta)/\theta$ encapsulates the difference between the pendulum and the ideal spring. When $f(\theta)$ is close to 1, the pendulum acts like an ideal spring; when $f(\theta)$ falls significantly below 1, the simple-harmonic approximation falls in accuracy. Having isolated the complexity into $f(\theta)$, the next step is to approximate $f(\theta)$ until the pendulum equation becomes easy to solve.

4.5.3 Adding discretization

The differential equation's nonlinearity is now represented by a changing $f(\theta)$. When change and complexity appear in the same sentence, pull out the discretization tool. In other words, replace the slowly changing $f(\theta)$ with a simpler, constant value.



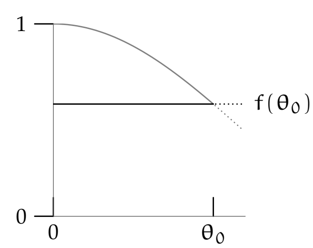
The simplest choice is to replace $f(\theta)$ with $f(0)$. Since $f(0) = 1$, the differential equation becomes

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0.$$

It is once again the ideal-spring equation, which produces a period independent of amplitude. So the simplest discretization $f(\theta) \longrightarrow f(0)$ is too crude to provide new information about how the period depends on amplitude.

▶ *What about discretizing using the other extreme of* θ*?*

The absolute pendulum angle $|\theta|$ lives in the range $[0, \theta_0]$. Since the first endpoint $\theta = 0$ was not a useful angle for discretizing, try the other endpoint θ_0.



In other words, replace the changing $f(\theta)$ not with $f(0)$ but with the slightly smaller constant $f(\theta_0)$. That change replaces $f(\theta)$ with a straight line, and turns the pendulum differential equation into

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta f(\theta_0) = 0.$$

▶ *Is this equation linear? What physical system does it describe?*

This equation is linear! Even better, it is familiar: It describes an ideal spring on a planet with slightly weaker gravity than earth's:

$$\frac{d^2\theta}{dt^2} + \frac{\overbrace{gf(\theta_0)}^{g_{\text{eff}}}}{l}\theta = 0,$$

where the gravity on the planet is $g_{\text{eff}} \equiv gf(\theta_0)$. Since an ideal spring has period $T = 2\pi\sqrt{l/g}$, this ideal spring has period

$$T = 2\pi\sqrt{\frac{l}{g_{\text{eff}}}} = 2\pi\sqrt{\frac{l}{g f(\theta_0)}}.$$

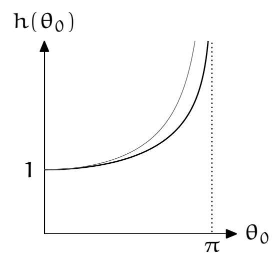
To compare this result with the ideal-spring period, rewrite it in dimensionless form using dimensionless quantities. One quantity, the amplitude θ_0, is already dimensionless. The period T is not dimensionless, but the dimensionless period $h(\theta_0)$ is defined as

$$h(\theta_0) \equiv \frac{T}{2\pi\sqrt{l/g}}.$$

The 2π in the definition makes the small-amplitude limit come out simple: $h(0) = 1$. With that definition for $h(\theta_0)$, the the discretization $f(\theta) \longrightarrow f(\theta_0)$ predicts

$$h(\theta_0) = \sqrt{\frac{l}{g f(\theta_0)}} \Big/ \sqrt{\frac{l}{g}} = f(\theta_0)^{-1/2}.$$

Since $f(\theta_0) = (\sin\theta_0)/\theta_0$, the dimensionless period becomes



$$h(\theta_0) = \left(\frac{\sin\theta_0}{\theta_0}\right)^{-1/2}.$$

This prediction (gray curve) matches the exact dimensionless period (black curve) quite well at small but nonzero amplitudes.

The comparison is easiest to make in that limit of small but nonzero amplitude θ_0. In that limit, the Taylor series for sine is

$$\sin\theta \approx \theta - \frac{\theta^3}{6},$$

so

$$\frac{\sin\theta_0}{\theta_0} \approx 1 - \frac{\theta^2}{6}.$$

Therefore

$$h(\theta_0) \approx \left(1 - \frac{\theta_0^2}{6}\right)^{-1/2}.$$

Since $\theta_0^2/6$ is even smaller than θ_0, which is itself small, the right side further simplifies using the binomial approximation (for small x):

$$(1+x)^{-1/2} \approx 1 - \frac{x}{2}.$$

Then the dimensionless period becomes

$$h(\theta_0) \approx \left(1 - \frac{\theta_0^2}{6}\right)^{-1/2} \approx 1 + \frac{\theta_0^2}{12}.$$

Putting back the dimensional quantities, the period is

$$T \approx 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{\theta_0^2}{12}\right).$$

▶ *Is this result an underestimate or an overestimate?*

The discretization approximation used the lowest possible effective gravity g_{eff}, namely its value at the endpoint $\theta = \theta_0$. Since weak gravity produces a long period, the approximation overestimates the period. Indeed, the exact coefficient of θ_0^2 is $1/16$ rather than $1/12$; see for example [26] for the following infinite series:

$$h(\theta_0) = 1 + \frac{1}{16}\theta_0^2 + \frac{11}{3072}\theta_0^4 + \cdots.$$

Problem 4.21 Slope revisited
Use the preceding result for $h(\theta_0)$ to check your conclusion in Problem 4.20 about the slope of $h(\theta_0)$ at $\theta_0 = 0$.

4.6 Summary and problems

Discretization turns calculus on its head. Whereas calculus analyzes a changing process by dividing it into ever finer intervals, discretization simplifies a changing process by lumping it into one unchanging process. Discretization turns curves into straight lines, so difficult integrals turn into rectangles, and mildly nonlinear differential equations turn into linear differential equations. Even though lumping sacrifices accuracy, it provides great simplicity.