Long-lasting learning

The theme of this book is how to understand new fields, whether the field is known generally but is new to you; or the field is new to everyone. In either case, certain ways of thinking promote understanding and long-term learning. This afterword illustrates these ways by using an example that has appeared twice in the book – the volume of a pyramid.

Remember nothing!

The volume is proportional to the height, because of the drilling-core argument. So $V \propto h$. But a dimensionally correct expression for the volume needs two additional lengths. They can come only from b^2. So

$$V \sim bh^2.$$

But what is the constant? It turns out to be 1/3.

Connect to other problems

Is that 3 in the denominator new information to remember? No! That piece of information also connects to other problems.

First, you can derive it by using special cases, which is the subject of **Section 7.1**.

Second, 3 is also the dimensionality of space. That fact is not a coincidence. Consider the simpler but analogous problem of the area of a triangle. Its area is

$$A = \frac{1}{2}bh.$$

The area has a similar form as the volume of the pyramid: A constant times a factor related to the base times the height. In two dimensions the constant is 1/2. So the 1/3 is likely to arise from the dimensionality of space.

That analysis makes the 3 easy to remember and thereby the whole formula for the volume.

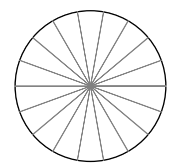
But there are two follow-up questions. The first is: Why does the dimensionality of space matter? The special-cases argument explains it because you need pyramids for each direction of space (I say no more for the moment until we do the special-cases argument in lecture!).

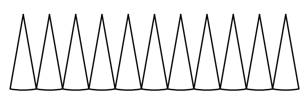
The second follow-up question is: Does the 3 occur in other problems and for the same reason? A related place is the volume of a sphere

$$V = \frac{4}{3}\pi r^3.$$

The ancient Greeks showed that the 3 in the 4/3 is the same 3 as in the pyramid volume. To explain their picture, I'll use method to find the area of a circle then use it to find the volume of a sphere.

Divide a circle into many pie wedges. To find its area, cut somewhere on the circumference and unroll it into this shape:





Each pie wedge is almost a triangle, so its area is $bh/2$, where the height h is approximately r. The sum of all the bases is the circumference $2\pi r$, so $A = 2\pi r \times r/2 = \pi r^2$.

Now do the same procedure with a sphere: Divide it into small pieces that are almost pyramids, then unfold it. The unfolded sphere has a base area of $4\pi r^2$, which is the surface area of the sphere. So the volume of all the mini pyramids is

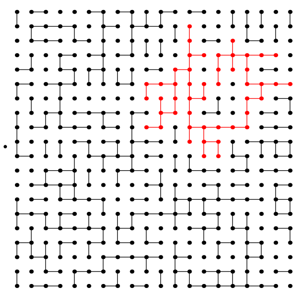
$$V = \frac{1}{3} \times \underbrace{\text{height}}_{r} \times \underbrace{basearea}_{4\pi r^2} = \frac{4}{3}\pi r^3.$$

Voilà! So, if you remember the volume of a sphere – and most of us have had it etched into our minds during our schooling – then you know that the volume of a pyramid contains a factor of 3 in the denominator.

Percolation model

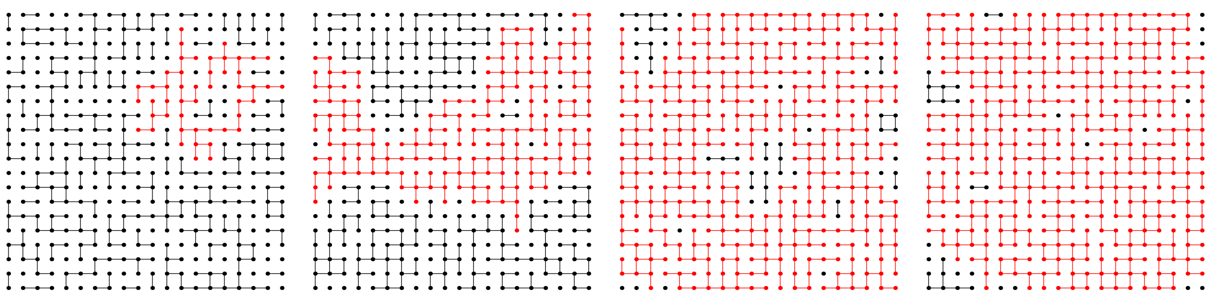
The moral of the preceding examples is to build connections. A physical illustration of this process is *percolation*. Imagine how oil diffuses through rock. The rock has pores through which oil moves from zone to zone. However, many pores are blocked by mineral deposits. How does the oil percolate through that kind of rock?

That question has led to an extensive mathematics research on the following idealized model. Imagine an infinite two-dimensional lattice. Now add bonds between neighbors (horizontal or vertical, not diagonal) with probability $p_{\mathtt{bond}}$. The figure shows an example of a finite subsection of a percolation lattice where $p_{\mathtt{bond}} = 0.4$. Its largest cluster – the largest set of points connected to each other – is marked in red, and contains 13 of the points.



cluster = 12%
$p = 0.40$

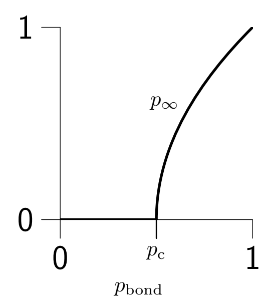
Here is what happens as $p_{\mathtt{bond}}$ increases from 0.40 to 0.50 to 0.55 to 0.60:



cluster = 12% $p = 0.40$ | cluster = 43% $p = 0.50$ | cluster = 87% $p = 0.55$ | cluster = 91% $p = 0.60$

The largest cluster occupies more and more of the lattice.

For an infinite lattice, a similar question is: What is the probability p_∞ of finding an infinite connected sublattice? That probability is zero until $p_{\mathtt{bond}}$ reaches a critical probability p_c. The critical probability depends on the topology (what kind of lattice and how many dimensions) – for the two-dimensional square lattice, $p_c = 1/2$ – but its existence is independent of topology. When $p_{\mathtt{bond}} > p_c$, the probability of a finding an infinite lattice becomes nonzero and eventually reaches 1.0.



An analogy to learning is that each lattice point (each dot) is a fact or formula, and each bond links two facts. For long-lasting learning, you want the facts to support each other via their connections. Let's say that you want the facts to become part of an infinite and therefore self-supporting lattice. However, if your textbooks or way of learning means that you just add more dots – learn just more facts – then you decrease p_{bond}, so you decrease the chance of an infinite clusters. If the analogy is more exact than I think it is, you might even eliminate infinite clusters altogether.

The opposite approach is to ensure that, with each fact, you create links to facts that you already know. In the percolation model, you add bonds between the dots in order to increase p_{bond}. A famous English writer gave the same advice about life that I am giving about learning:

> Only connect! That was the whole of her sermon. . . Live in fragments no longer! [E. M. Forster, *Howard's End*]

The ways of reasoning presented in this book offer some ways to build those connections. Bon voyage as you learn and discover new ideas and the links between them!