5. $V$: 30–60 gallons;
6. $f_{\text{other}}$: 1.5–3;
7. $f_{\text{imported}}$: 0.3–0.8.

▶ *What is the resulting plausible range for the oil imports?*

Now combine the ranges using the method we used for the area of a sheet of A4 paper. That method produces the following plausible range:

$$1.0 \ldots 3.1 \ldots 9.6 \cdot 10^9 \text{ barrels/year.}$$

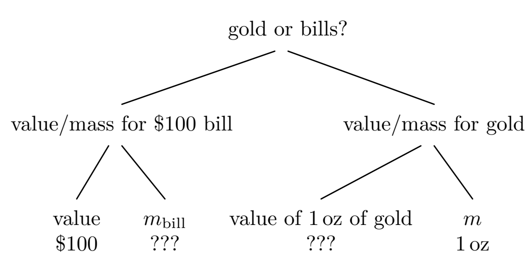
Compare this range to the range for the off-the-cuff guess $10^7 \ldots 10^{12}$ barrels/yr. That range spanned a factor of $10^5$ whereas the improved range spans a mere factor of 10 – thanks to divide-and-conquer reasoning.

## 2.6 Example 3: Gold or bills?

The chapter's final estimation example is dedicated to readers who forgo careers in the financial industry for less lucrative careers in teaching and research:

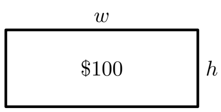
▶ *Having broken into a bank vault, should we take the $100 bills or the gold?*

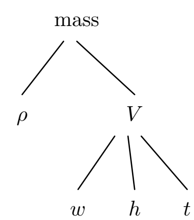
The answer depends partly on the ease and costs of fencing the loot – an analysis beyond the scope of this book. But within our scope is the following question: Which choice lets us carry out the most money? Our carrying capacity is limited by weight and volume. In this analysis, I assume that the lowest limit comes from weight (or mass). The mass subdivides into two subproblems – the value per mass for $100 bills and the value per mass for gold – each of which subdivides into two subproblems:




Two leaves have defined values: the value of a $100 bill and the mass of 1 oz (1 ounce) of gold. The two other leaves need divide-and-conquer estimates. In the first round of analysis I make point estimates; in the second round, I account for uncertainty by using the plausible ranges of **Section 2.5**.

The value of gold is now (2008), I vaguely remember, around \$800/oz. As a rough check on the value – for example, should it be \$80/oz or \$8000/oz? – here is a historical method. In 1945, at the end of World War 2, the British empire had exhausted its resources while the United States became the world's leading economic power. The gold standard was accordingly redefined in terms of the dollar: \$35 would be the value of 1 oz of gold. Since then, inflation has probably devalued the dollar by a factor of 10 or more, so gold should be worth around \$350/oz. So my vague memory of \$800/oz seems reasonable.




For the \$100 bill, its mass breaks into density ($\rho$) times volume ($V$), and volume breaks into width ($w$) times height ($h$) times thickness ($t$). To estimate the height and width, I could lay down a ruler or just find a \$1 bill – all US bills are the same size – and eyeball its dimensions. A \$1 bill seems to be few inches high and 6 in wide. In metric units those dimensions are $h \sim 6\,\text{cm}$ and $w \sim 15\,\text{cm}$. [To improve your judgment of size, first make guesses; then, if you feel unsure, check the guess using a ruler to check. With practice, your need for the ruler will decrease and your confidence and accuracy will increase.]

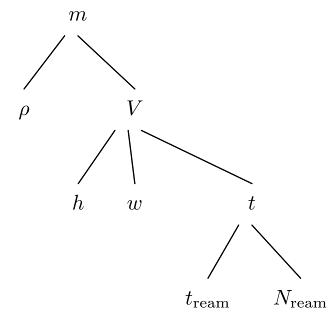
The thickness, alas, is not easy to estimate with eyeball or ruler. Is the thickness 1 mm or 0.1 mm or 0.01 mm? I have almost no experience with such small lengths so my eye does not not help much. My ruler is calibrated in steps of 1 mm, from which I see that a piece of paper is significantly smaller than 1 mm, but I cannot easily see how much smaller.

An accurate divide-and-conquer estimate, we learned in **Section 2.5**, depends on replacing a vaguely understood quantity with accurately known quantities. Therefore to estimate the thickness accurately, I connect it to familiar quantities. Bills are made from paper, a ubiquitous substance (despite hype about the paperless office). Indeed, a ream of printer paper is just around the corner. The thickness of the ream and the number of sheets that it contains determines the thickness of one sheet:

$$t = \frac{t_{ream}}{N_{ream}}.$$

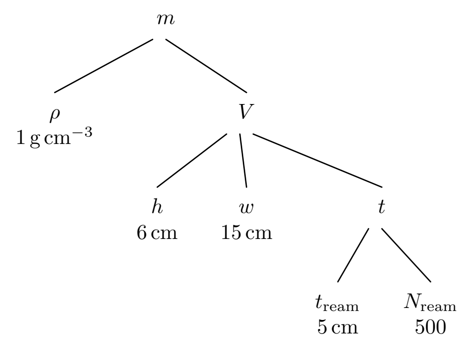
You could call this approach 'multiply and conquer'. The general lesson for accurate estimation is that values well below our experience need to be magnified, and values well above our experience need to be shrunk.

The magnification argument adds one level to the tree and replaces one leaf with two leaves on the new level. Two of the five leaf nodes are already estimated. A ream contains 500 sheets ($N_{ream} = 500$) and has a thickness of roughly 2 in or 5 cm.




▶ *What is your estimate for ρ, the density of a $100 bill?*

The only missing leaf value is ρ, the density of a $100 bill. Connect this value to what you already know such as the densities of familiar substances. Bills are made of paper, whose density is hard to guess directly. However, paper is made of wood, whose density is easy to guess! Wood barely floats so its density is roughly that of water: $1\,\text{g}\,\text{cm}^{-3}$. Therefore the density of a $100 bill is roughly $1\,\text{g}\,\text{cm}^{-3}$.



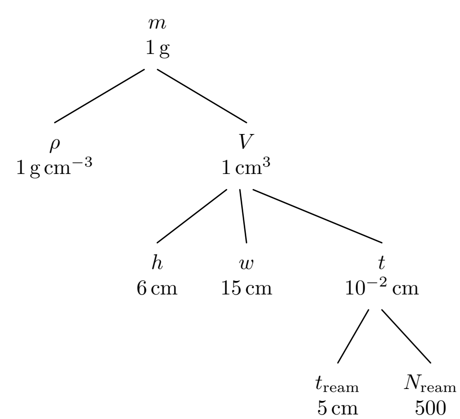

Now propagate the leaf values upward. The thickness of a bill is roughly $10^{-2}$ cm, so the volume of a bill is roughly

$$V \sim 6\,\text{cm} \times 15\,\text{cm} \times 10^{-2}\,\text{cm} \sim 1\,\text{cm}^3.$$

The mass is therefore

$$m \sim 1\,\text{cm}^3 \times 1\,\text{g}\,\text{cm}^{-3} \sim 1\,\text{g}.$$

and the value per mass of a $100 bill is therefore $100/g. How simple!




To choose between the bills and gold, compare that value to the value per mass of gold. Unfortunately the price of gold is usually quoted in dollars per ounce rather than dollars per gram, so my vague memory of $800/oz needs to be converted into metric units. One ounce is roughly 28 g; if the price of gold were $840/oz, the arithmetic is simple enough to do mentally, and produces $30/g. The exact division produces the slightly lower figure of $28/g. Our conclusion: In the bank vault first collect as many \$100 bills as we can carry. If we have spare capacity, collect the \$50 bills, the gold, and then the \$20 bills.

This order depends on the accuracy of the point estimates and would change if the estimates are significantly inaccurate. How accurate are they? To analyze the accuracy I will give plausible ranges for the leaf nodes and then propagate them upward to obtain plausible ranges for the value per mass of bills and gold.

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**Problem 2.10 Your plausible ranges**
What are your plausible ranges for the five leaf quantities $t_{ream}$, $N_{ream}$, $w$, $h$, and $\rho$? Propagate them upward to get plausible ranges for the interior nodes including for the root node $m$.

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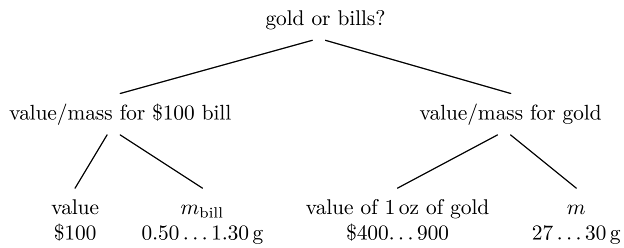
Here are my ranges along with a few notes on how I estimated a few of the ranges:

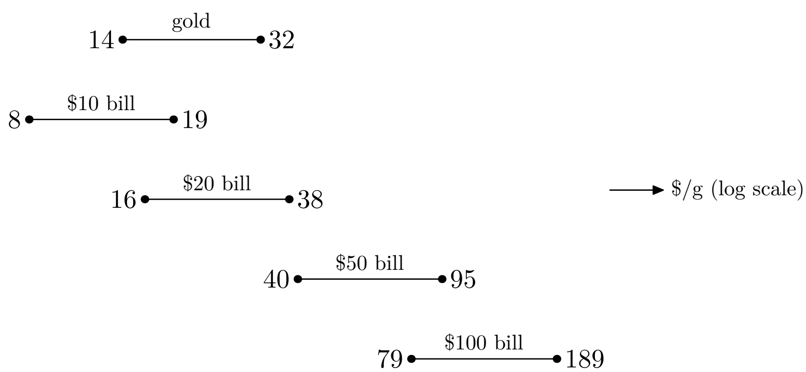
1. thickness of a ream, $t_{ream}$: $4 \ldots 6$ cm.
2. number of sheets in a ream, $N_{ream}$: 500. I'm almost certain that I remember this value correctly, but to be certain I confirmed it by looking at a label on a fresh ream.
3. width of a bill, $w$: $10 \ldots 20$ cm. A reasonable length estimate seemed to be 6 in but I could give or take a couple inches. In metric units, $4 \ldots 8$ in becomes (roughly) $10 \ldots 20$ cm.
4. height of a bill, $h$: $5 \ldots 7$ cm.
5. density of a bill, $\rho$: $0.8 \ldots 1.2\,\mathrm{g\,cm^{-3}}$. The argument for $\rho = 1\,\mathrm{g\,cm^{-3}}$ – that a bill is made from paper and paper is made from wood – seems reasonable. However, the processing steps may reduce or increase the density slightly.

Now propagate these ranges upward. The plausible range for $t$ becomes $0.8 \ldots 1.2 \cdot 10^{-2}$ cm. The plausible range for the volume V becomes $0.53 \ldots 1.27\,\mathrm{cm}^3$. The plausible range for the mass $m$ becomes $0.50 \ldots 1.30$ g. The plausible range for the value per mass is \$79. . . 189/g (with a midpoint of \$122/g).




The next estimate is the value per mass of gold. I can be as accurate as I want in converting from ounces to grams. But I'll be lazy and try to remember the value while including uncertainty to reflect the fallibility of memory; let's say that $1\,\text{oz} = 27\ldots30\,\text{g}$. This range spans only a factor of 1.1, but the value of an ounce of gold will have a wider plausible range (except for those who often deal with financial markets). My range is $400. . . 900. The mass and value ranges combine to give $14. . . 32/g as the range for gold.

Here is a picture comparing the range for gold with the ranges for US currency denominations:




Looking at the locations of these ranges and overlaps among them, I am confident that the $100 bills are worth more (per mass) than gold. I am reasonably confident that $50 bills are worth more than gold, undecided about $20 bills, and reasonably confident that $10 bills are worth less than gold.

## 2.7 Example 4: The UNIX philosophy

The preceding examples illustrate how divide and conquer enables accurate estimates. An example remote from estimation – the design principles of the UNIX operating system – illustrates the generality of this tool.

UNIX and its close cousins such as GNU/Linux operate devices as small as cellular telephones and as large as supercomputers cooled by liquid nitrogen. They constitute the world's most portable operating system. Its success derives not from marketing – the most succesful variant, GNU/Linux,