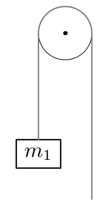
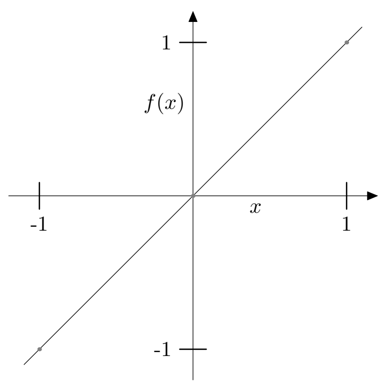
To guess $f(x)$, where $x = G_1$, try special cases. First imagine that m_1 becomes huge. A quantity with mass cannot be huge on its own, however. Here huge means *huge relative to* m_2, whereupon $x \approx 1$. In this thought experiment, m_1 falls as if there were no m_2 so $a = -g$. Here we've chosen a sign convention with positive acceleration being upward. If m_2 is huge relative to m_1, which means $x = -1$, then m_2 falls like a stone pulling m_1 upward with acceleration $a = g$. A third limiting case is $m_1 = m_2$ or $x = 0$, whereupon the masses are in equilibrium so $a = 0$.



Here is a plot of our knowledge of f:



The simplest conjecture – an educated guess – is that $f(x) = x$. Then we have our result:

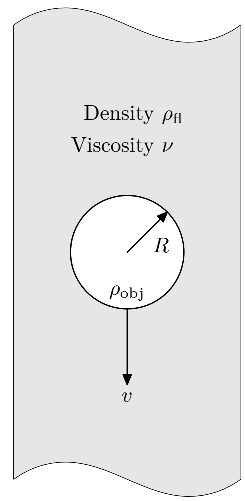
$$\frac{a}{g} = \frac{m_1 - m_2}{m_1 + m_2}.$$

Look how simple the result is when derived in a symmetric, dimensionless form using special cases!

7.3 Drag

Pendulum motion is not a horrible enough problem to show the full benefit of dimensional analysis. Instead try fluid mechanics – a subject notorious for its mathematical and physical complexity; Chandrasekhar's books [6, 7] or the classic textbook of Lamb [19] show that the mathematics is not for the faint of heart.

The next examples illustrate two extremes of fluid flow: oozing and turbulent. An example of oozing flow is ions transporting charge in seawater (**Section 7.3.6**). An example of turbulent flow is a raindrop falling from the sky after condensing out of a cloud (**Section 7.3.7**).



To find the terminal velocity, solve the partial-differential equations of fluid mechanics for the incompressible flow of a Newtonian fluid:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{v}, \qquad \text{(3 eqns)}$$

$$\nabla\cdot\mathbf{v} = 0. \qquad \text{(1 eqn)}$$

Here $\mathbf{v}$ is the fluid velocity, ρ is the fluid density, ν is the kinematic viscosity, and p is the pressure. The first equation is a vector shorthand for three equations, so the full system is four equations.

All the equations are partial-differential equations and three are nonlinear. Worse, they are coupled: Quantities appear in more than one equation. So we have to solve a system of coupled, nonlinear, partial-differential equations. This solution must satisfy boundary conditions imposed by the marble or raindrop. As the object moves, the boundary conditions change. So until you know how the object moves, you do not know the boundary conditions. Until you know the boundary conditions, you cannot find the motion of the fluid or of the object. This coupling between the boundary conditions and solution compounds the difficulty of the problem. It requires that you solve the equations and the boundary conditions together. If you ever get there, then you take the limit $t \to \infty$ to find the terminal velocity.

Sleep easy! I wrote out the Navier–Stokes equations only to scare you into using dimensional analysis and special-cases reasoning. The approximate approach is easier than solving nonlinear partial-differential equations.

7.3.1 Naive dimensional analysis

To use dimensional analysis, follow the usual steps: Choose relevant variables, form dimensionless groups from them, and solve for the terminal velocity. In choosing quantities, do not forget to include the variable for which you are solving, which here is v. To decide on the other quantities, split them into three categories (divide and conquer):

1. characteristics of the fluid,
2. characteristics of the object, and
3. characteristics of whatever makes the object fall.

The last category is the easiest to think about, so deal with it first. Gravity makes the object fall, so g is on the list.

Consider next the characteristics of the object. Its velocity, as the quantity for which we are solving, is already on the list. Its mass m affects the terminal velocity: A feather falls more slowly than a rock does. Its radius r probably affects the terminal velocity. Instead of listing r and m together, remix them and use r and ρ_{obj}. The two alternatives r and m or r and ρ_{obj} provide the same information as long as the object is uniform: You can compute ρ_{obj} from m and r and can compute m from ρ_{obj} and r.

Choose the preferable pair by looking ahead in the derivation. The relevant properties of the fluid include its density ρ_{fl}. If the list also includes ρ_{obj}, then the results might contain pleasing dimensionless ratios such as ρ_{obj}/ρ_{fl} (a dimensionless group!). The ratio ρ_{obj}/ρ_{fl} has a more obvious physical interpretation than a combination such as $m/\rho_{fl}r^3$, which, except for a dimensionless constant, is more obscurely the ratio of object and fluid densities. So choose ρ_{obj} and r over m and r.

Scaling arguments also favor the pair ρ_{obj} and r. In a scaling argument you imagine varying, say, a size. Size, like heat, is an extensive quantity: a quantity related to amount of stuff. When you vary the size, you want as few other variables as possible to change so that those changes do not obscure the effect of changing size. Therefore, whenever possible replace extensive quantities with **intensive quantities** like temperature or density. The pair m and r contains two extensive quantities, whereas the preferable pair ρ_{obj} and r contains only one extensive quantity.

Now consider properties of the fluid. Its density ρ_{fl} affects the terminal velocity. Perhaps its viscosity is also relevant. Viscosity measures the tendency of a fluid to reduce velocity differences in the flow. You can observe an analog of viscosity in traffic flow on a multilane highway. If one lane moves much faster than another, drivers switch from the slower to the faster lane, eventually slowing down the faster lane. Local decisions of the drivers reduce the velocity gradient. Similarly, molecular motion (in a gas) or collisions (in a fluid) transports speed (really, momentum) from fast- to slow-flowing regions. This transport reduces the velocity difference between the regions. Oozier (more viscous) fluids probably produce more drag than thin fluids do. So viscosity belongs on the list of relevant variables.

Fluid mechanicians have defined two viscosities: dynamic viscosity η and kinematic viscosity ν. [Sadly, we could not use the mellifluous term *fluid mechanics* to signify a host of physicists agonizing over the equations of fluid mechanics; it would not distinguish the toilers from their toil.] The two viscosities are related by $\eta = \rho_{fl}\nu$. *Life in Moving Fluids* [36, pp. 23–25] discusses the two types of viscosity in detail. For the analysis of drag force, you need to know only that viscous forces are proportional to viscosity. Which viscosity should we use? Dynamic viscosity hides ρ_{fl} inside the product $\nu\rho_{fl}$; a ratio of ρ_{obj} and η then looks less dimensionless than it is because ρ_{obj}'s partner ρ_{fl} is buried inside η. Therefore the kinematic viscosity ν usually gives the more insightful results. Summarizing the discussion, the table lists the variables by category.

Var	Dim	What
ν	L^2T^{-1}	kinematic viscosity
ρ_{fl}	ML^{-3}	fluid density
r	L	object radius
v	LT^{-1}	terminal velocity
ρ_{obj}	ML^{-3}	object density
g	LT^{-2}	gravity

The next step is to find dimensionless groups. The Buckingham Pi theorem (**Section 6.6**) says that the six variables and three independent dimensions result in three dimensionless groups.

Before finding the groups, consider the consequences of three groups. Three?! Three dimensionless groups produce this form for the terminal velocity v:

$$\text{group with } v = f(\text{other group 1}, \text{other group 2}).$$

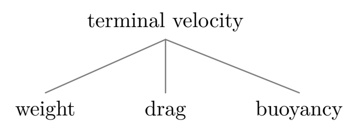
To deduce the properties of f requires physics knowledge. However, studying a two-variable function is onerous. A function of one variable is represented by a curve and can be graphed on a sheet of paper. A function of two variables is represented by a surface. For a complete picture it needs three-dimensional paper (do you have any?); or you can graph many slices of it on regular two-dimensional paper. Neither choice is appealing. This brute-force approach to the terminal velocity produces too many dimensionless groups.

If you simplify only after you reach the complicated form

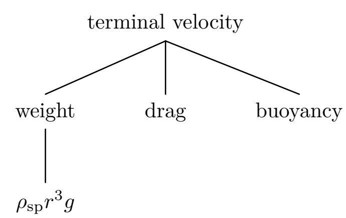
$$\text{group with } v = f(\text{other group 1}, \text{other group 2}),$$

you carry baggage that you eventually discard. When going on holiday to the Caribbean, why pack skis that you never use but just cart around everywhere? Instead, at the beginning of the analysis, incorporate the physics knowledge. That way you simplify the remainder of the derivation. To follow this strategy of packing light – of packing only what you need – consider the physics of terminal velocity in order to make simplifications now.

7.3.2 Simpler approach



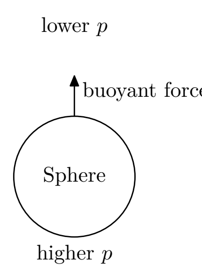
The adjective *terminal* in the phrase 'terminal velocity' hints at the physics that determines the velocity. Here 'terminal' is used in its sense of final, as in after an infinite time. It indicates that the velocity has become constant, which happens only when no net force acts on the marble. This line of thought suggests that we imagine the forces acting on the object: gravity, buoyancy, and drag. The terminal velocity is velocity at which the drag, gravitational, and buoyant forces combine to make zero net force. Divide-and-conquer reasoning splits the terminal-velocity problem into three simpler problems.



The gravitational force, also known as the weight, is mg. Instead of m we use $(4\pi/3)\rho_{obj}r^3$ – for the same reasons that we listed ρ_{obj} instead of m in the table of variables – and happily ignore the factor of $4\pi/3$. With those choices, the weight is

$$F_g \sim \rho_{obj} r^3 g.$$

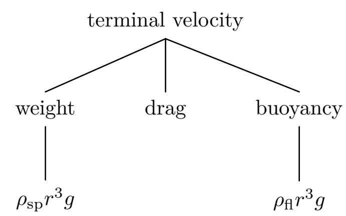
The figure shows the roadmap updated with this information.



The remaining pieces are drag and buoyancy. Buoyancy is easier, so do it first (the principle of maximal laziness). It is an upward force that results because gravity affects the pressure in a fluid. The pressure increases according to $p = p_0 + \rho_{fl} g h$, where h is the depth and p_0 is the pressure at zero depth (which can be taken to be at any level in the fluid). The pressure difference between the top and bottom of the object, which are separated by a distance $\sim r$, is $\Delta p \sim \rho_{fl} g r$. Pressure is force per area, and the pressure difference acts over an area $A \sim r^2$. Therefore the buoyant force created by the pressure difference is

$$F_b \sim A\Delta p \sim \rho_{fl} r^3 g.$$

As a check on this result, Archimedes's principle says that the buoyant force is the 'weight of fluid displaced'. This weight is



$$\overbrace{\rho_{fl} \underbrace{\frac{4\pi}{3}\pi r^3}_{volume}}^{mass} g.$$

Except for the factor of $4\pi/3$, it matches the buoyant force so Archimedes's principle confirms our estimate for F_b. That result updates the roadmap. The main unexplored branch is the drag force, which we solve using dimensional analysis.

7.3.3 Dimensional analysis for the drag force

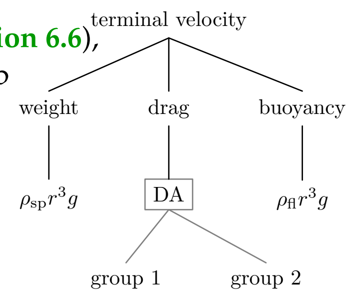
The weight and buoyancy were solvable without dimensional analysis, but we still need to use dimensional analysis to find the drag force. The purpose of breaking the problem into parts was to simplify this dimensional analysis relative to the brute-force approach in **Section 7.3.1**. Let's see how the list of variables changes when computing the drag force rather than the terminal velocity. The drag force F_d has to join the list: not a promising beginning when trying to eliminate variables. Worse, the terminal velocity v remains on the list, even though we are no longer computing it, because the drag force depends on the velocity of the object.

However, all is not lost. The drag force has no idea what is inside the sphere. Picture the fluid as a huge computer that implements the laws of fluid dynamics. From the viewpoint of this computer, the parameters v and r are the only relevant attributes of a moving sphere. What lies underneath the surface does not affect the fluid flow: Drag is only skin deep. The computer can determine the flow (if it has tremendous processing power) without knowing the sphere's density ρ_{obj}, which means it vanishes from the list. Progress!

Var	Dim	What
F_d	MLT^{-2}	drag force
ν	L^2T^{-1}	kinematic viscosity
ρ_{fl}	ML^{-3}	fluid density
r	L	object radius
v	LT^{-1}	terminal velocity

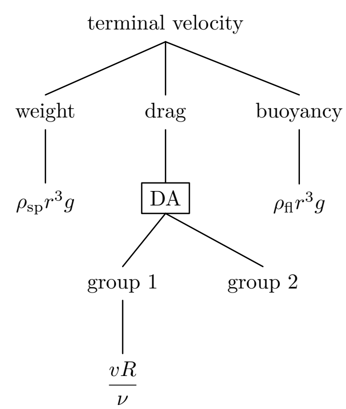
Now consider the characteristics of the fluid. The fluid supercomputer still needs the density and viscosity of the fluid to determine how the pieces of fluid move in response to the object's motion. So ρ_{fl} and ν remain on the list. What about gravity? It causes the object to fall, so it is responsible for the terminal velocity v. However, the fluid supercomputer does not care how the object acquired this velocity; it cares only what the velocity is. So g vanishes from the list. The updated tabled shows the new, shorter list.

The five variables in the list are composed of three basic dimensions. From the Buckingham Pi theorem (**Section 6.6**), we expect two dimensionless groups. We find one group by dividing and conquering. The list already includes a velocity (the terminal velocity). If we can concoct another quantity V with dimensions of velocity, then v/V is a dimensionless group. The viscosity ν is almost a velocity. It contains one more power of length than velocity does. Dividing by r eliminates the extra length: $V \equiv \nu/r$. A dimensionless group is then



$$G_1 \equiv \frac{v}{V} = \frac{vr}{\nu}.$$

Our knowledge, including this group, is shown in the figure. This group is so important that it has a name, the **Reynolds number**, which is abbreviated Re. It is important because it is a *dimensionless* measure of flow speed. The velocity, because it contains dimensions, cannot distinguish fast from slow flows. For example, $1000\,\mathrm{m\,s^{-1}}$ is slow for a planet, whose speeds are typically tens of kilometers per second, but fast for a pedestrian. When you hear that a quantity is small, fast, large, expensive, or almost any adjective, your first reaction should be to ask, 'compared to what?' Such a comparison suggests dividing v by another velocity; then we get a dimensionless quantity that is proportional to v. The result of this division is the Reynolds number.



Low values of Re indicate slow, viscous flow (cold honey oozing out of a jar). High values indicate turbulent flow (a jet flying at 600 mph). The excellent *Life in Moving Fluids* [36] discusses many more dimensionless ratios that arise in fluid mechanics.

The Reynolds number looks lonely in the map. To give it company, find a second dimensionless group. The drag force is absent from the first group so it must live in the second; otherwise we cannot solve for the drag force.

Instead of dreaming up the dimensionless group in one lucky guess, we construct it in steps (divide-and-conquer reasoning). Examine the variables in the table, dimension by dimension. Only two (F_d and ρ_{fl}) contain mass, so both or neither appear in the group. Because F_d has to appear, ρ_{fl} must also appear. Each variable contains a first power of mass, so the group contains the ratio F_d/ρ_{fl}. A simple choice is

$$G_2 \propto \frac{F_d}{\rho_{fl}}.$$

The dimensions of F_d/ρ_{fl} are L^4T^{-2}, which is the square of L^2T^{-1}. Fortune smiles on us, for L^2T^{-1} are the dimensions of ν. So

$$\frac{F_d}{\rho_{fl}\nu^2}$$

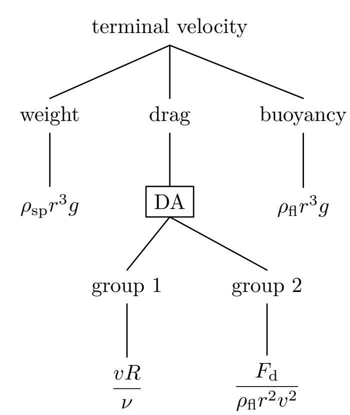
is a dimensionless group.

This choice, although valid, has a defect: It contains ν, which already belongs to the first group (the Reynolds number). Of all the variables in the problem, ν is the one most likely to be found irrelevant based on a physical argument (as will happen in **Section 7.3.7**, when we specialize to high-speed flow. If ν appears in two groups, eliminating it requires recombining the two groups into one that does not contain ν. However, if ν appears in only one group, then eliminating it is simple: eliminate that group. Simpler mathematics – eliminating a group rather than remixing two groups to get one group – requires simpler physical reasoning. Therefore, isolate ν in one group if possible.

To remove ν from the proposed group $F_d/\rho_{fl}\nu^2$ notice that the product of two dimensionless groups is also dimensionless. The first group contains ν^{-1} and the proposed group contains ν^{-2}, so the ratio

$$\frac{\text{group proposed}}{(\text{first group})^2} = \frac{F_d}{\rho_{fl} r^2 v^2}$$

is not only dimensionless but it also does not contain ν. So the analysis will be easy to modify when we try to eliminate ν. With this revised second group, our knowledge is now shown in this figure:



This group, unlike the the proposal $F_d/\rho_{fl}\nu^2$, has a plausible physical interpretation. Imagine that the sphere travels a distance l, and use l to multiply the group by unity:

$$\underbrace{\frac{F_d}{\rho_{fl} r^2 v^2}}_{\textit{group 1}} \times \underbrace{\frac{l}{l}}_{1} = \frac{F_d l}{\rho_{fl} l r^2 v^2}.$$

The numerator is the work done against the drag force over the distance l. The denominator is also an energy. To interpret it, examine its parts (divide and conquer). The product lr^2 is, except for a dimensionless constant, the volume of fluid swept out by the object. So $\rho_{fl} lr^2$ is, except for a constant, the mass of fluid shoved aside by the object. The object moves fluid with a velocity comparable to v, so it imparts to the fluid a kinetic energy

$$E_K \sim \rho_{fl} lr^2 v^2.$$

Thus the ratio, and hence the group, has the following interpretation:

$$\frac{\text{work done against drag}}{\text{kinetic energy imparted to the fluid}}.$$

In highly dissipative flows, when energy is burned directly up by viscosity, the numerator is much larger than the denominator, so this ratio (which will turn out to measure drag) is much greater than 1. In highly streamlined flows (a jet wing), the the work done against drag is small because the fluid returns most of the imparted kinetic energy to the object. So in the ratio, the numerator will be small compared to the denominator.

To solve for F_d, which is contained in G_2, use the form $G_2 = f(G_1)$, which becomes

$$\frac{F_d}{\rho_{fl} r^2 v^2} = f\left(\frac{vr}{\nu}\right).$$

The drag force is then

$$F_d = \rho_{fl} r^2 v^2 \, f\left(\frac{vr}{\nu}\right).$$

The function f is a dimensionless function: Its argument is dimensionless and it returns a dimensionless number. It is also a universal function. The same f applies to spheres of any size, in a fluid of any viscosity or density! Although f depends on r, ρ_{fl}, ν, and v, it depends on them only through one combination, the Reynolds number. A function of one variable is easier to study than is a function of four variables:

> A good table of functions of one variable may require a page; that of a function of two variables a volume; that of a function of three variables a bookcase; and that of a function of four variables a library.
>
> —Harold Jeffreys [25, p. 82]

Dimensional analysis cannot tell us the form of f. To learn its form, we specialize to two special cases:

1. viscous, low-speed flow ($\mathrm{Re} \ll 1$), the subject of **Section 7.3.4**; and
2. turbulent, high-speed flow ($\mathrm{Re} \gg 1$), the subject of **Section 7.3.7**.

7.3.4 Viscous limit

As an example of the low-speed limit, consider a marble falling in vegetable oil or glycerin. You may wonder how often marbles fall in oil, and why we bother with this example. The short answer to the first question is 'not often'. However, the same physics that determines the fall of marbles in oil also determines, for example, the behavior of fog droplets in air, of bacteria swimming in water [26], or of oil drops in the Millikan oil-drop experiment. The marble problem not only illustrates the physical principles, but also we can check our results with a home experiment.

In slow, viscous flows, the drag force comes directly from – surprise! – viscous forces. These forces are proportional to viscosity because viscosity is the constant of proportionality in the definition of the viscous force. Therefore

$$F_d \propto \nu.$$

The viscosity appears exactly once in the drag result, repeated here:

$$F_d = \rho_{fl} r^2 v^2 \, f\left(\frac{vr}{\nu}\right).$$

To flip ν into the numerator and make $F_d \propto \nu$, the function f must have the form $f(x) \sim 1/x$. With this $f(x)$ the result is

$$F_d \sim \rho_{fl} r^2 v^2 \, \frac{\nu}{vr} = \rho_{fl} \nu v.$$

Dimensional analysis alone is insufficient to compute the missing magic dimensionless constant. A fluid mechanician must do a messy and difficult calculation. Her burden is light now that we have worked out the solution except for this one constant. The British mathematician Stokes, the first to derive its value, found that

$$F_d = 6\pi \rho_{fl} \nu v r.$$

In honor of Stokes, this result is called Stokes drag.

Let's sanity check the result. Large or fast marbles should feel a lot of drag, so r and v should be in the numerator. Viscous fluids should produce a lot of drag, so ν should be the numerator. The proposed drag force passes these tests. The correct location of the density – in the numerator or denominator – is hard to judge.

You can make an educated judgment by studying the Navier–Stokes equations. In those equations, when ν is 'small' (small compared to what?) then the $(\mathbf{v}\cdot\nabla)\mathbf{v}$ term, which contains two powers of ν, becomes tiny compared to the viscous term $\nu\nabla^2\mathbf{v}$, which contains only one power of ν. The second-order term arises from the inertia of the fluid, so this term's being small says that the oozing marble does not experience inertial effects. So perhaps ρ_{fl}, which represents the inertia of the fluid, should not appear in the Stokes drag. On the other hand, viscous forces are proportional to the *dynamic* viscosity $\eta = \rho_{fl}\nu$, so ρ_{fl} should appear even if inertia is unimportant. The Stokes drag passes this test. Using the dynamic instead of kinematic viscosity, the Stokes drag is

$$F_d = 6\pi\eta v r,$$

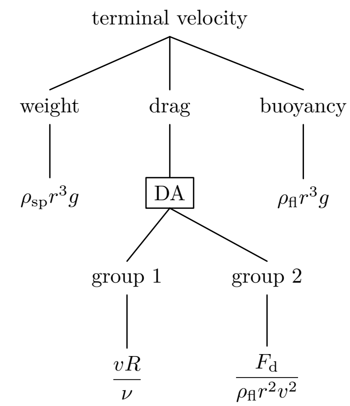
often a convenient form because many tables list η rather than ν.

This factor of 6π comes from doing honest calculations. Here, it comes from solving the Navier–Stokes equations. In this book we wish to teach you how not to suffer, so we do not solve such equations. We usually quote the factor from honest calculation to show you how accurate (or sloppy) the approximations are. The factor is often near unity, although not in this case where it is roughly 20! In fancy talk, it is usually 'of order unity'. Such a number suits our neural hardware: It is easy to remember and to use. Knowing the approximate derivation and remembering this one number, you reconstruct the exact result without solving difficult equations.

Now use the Stokes drag to estimate the terminal velocity in the special case of low Reynolds number.

7.3.5 Terminal velocity for low Reynolds number

Having assembled all the pieces in the roadmap, we now return to the original problem of finding the terminal velocity. Since no net force acts on the marble (the definition of terminal velocity), the drag force plus the buoyant force equals the weight:



$$\underbrace{\nu\rho_{fl}vr}_{F_d} + \underbrace{\rho_{fl}gr^3}_{F_b} \sim \underbrace{\rho_{obj}gr^3}_{F_g}.$$

After rearranging:

$$\nu\rho_{fl}vr \sim (\rho_{obj} - \rho_{fl})gr^3.$$

The terminal velocity is then

$$v \sim \frac{gr^2}{\nu}\left(\frac{\rho_{obj}}{\rho_{fl}} - 1\right).$$

In terms of the dynamic viscosity η, it is

$$v \sim \frac{gr^2}{\eta}(\rho_{obj} - \rho_{fl}).$$

This version, instead of having the dimensionless factor $\rho_{obj}/\rho_{fl} - 1$ that appears in the version with kinematic viscosity, has a dimensional $\rho_{obj} - \rho_{fl}$ factor. Although it is less aesthetic, it is often more convenient because tables often list dynamic viscosity η rather than kinematic viscosity ν.

We can increase our confidence in this expression by checking whether the correct variables are upstairs (a picturesque way to say 'in the numerator') and downstairs (in the denominator). Denser marbles should fall faster than less dense marbles, so ρ_{obj} should live upstairs. Gravity accelerates marbles, so g should live upstairs. Viscosity slows marbles, so ν should live downstairs. The terminal velocity passes these tests. We therefore have more confidence in our result, although the tests did not check the location of r or any exponents: For example, should ν appear as ν^2? Who knows, but if viscosity matters, it mostly appears as a square root or as a first power.

To check r, imagine a large marble. It will experience a lot of drag and fall slowly, so r should appear downstairs. However, large marbles are also heavy and fall rapidly, which suggests that r should appear upstairs. Which effect wins is not obvious, although after you have experience with these problems, you can make an educated guess: weight scales as r^3, a rapidly rising function r, whereas drag is probably proportional to a lower power of r. Weight usually wins such contents, as it does here, leaving r upstairs. So the terminal velocity also passes the r test.

Let's look at the dimensionless ratio in parentheses: $\rho_{obj}/\rho_{fl} - 1$. Without buoyancy the -1 disappears, and the terminal velocity would be

$$v \propto g\,\frac{\rho_{obj}}{\rho_{fl}}.$$

We retain the g in the proportionality for the following reason: The true solution returns if we replace g by an effective gravity g' where

$$g' \equiv g\left(1 - \frac{\rho_{fl}}{\rho_{obj}}\right).$$

So, one way to incorporate the effect of the buoyant force is to solve the problem without buoyancy but with the reduced g.

Check this replacement in two limiting cases: $\rho_{fl} = 0$ and $\rho_{fl} = \rho_{obj}$. When $\rho_{obj} = \rho_{fl}$ gravity vanishes: People, whose density is close to the density of water, barely float in swimming pools. Then g' should be zero. When $\rho_{fl} = 0$, buoyancy vanishes and gravity retains its full effect. So g' should equal g. The effective gravity definition satisfies both tests. Between these two limits, the effective g should vary linearly with ρ_{fl} because buoyancy and weight superpose linearly in their effect on the object. The effective g passes this test as well.

Another test is to imagine $\rho_{fl} > \rho_{obj}$. Then the relation correctly predicts that g' is negative: helium balloons rise. This alternative to using buoyancy explicitly is often useful. If, for example you forget to include buoyancy (which happened in the first draft of this chapter), you can correct the results later by replacing g with the g'.

If we carry forward the constants of proportionality, starting with the magic 6π in the Stokes drag and including the $4\pi/3$ that belongs in the weight, we find

$$v \sim \frac{2}{9}\,\frac{gr^2}{\nu}\left(\frac{\rho_{obj}}{\rho_{fl}} - 1\right).$$

7.3.6 Conductivity of seawater