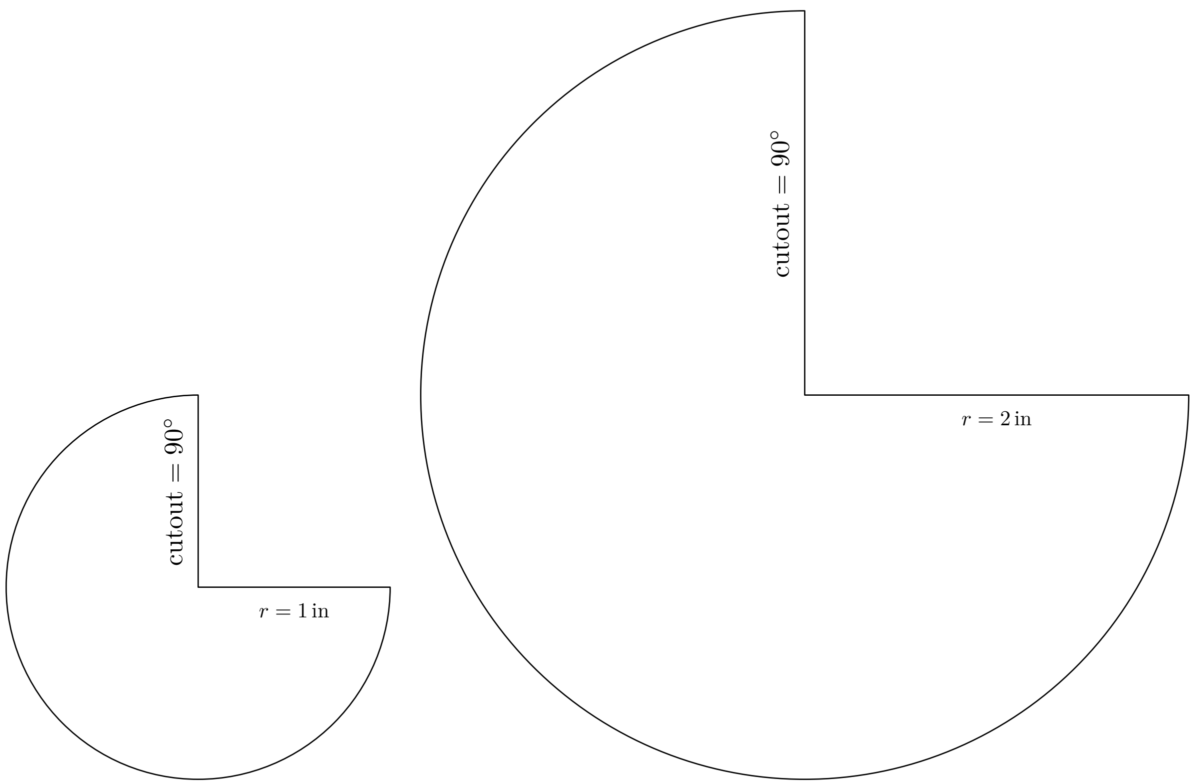
The moral of this section is: *When there is change, look for what does not change.* That quantity becomes a new abstraction (**Chapter 3**), so looking for invariants is a recipe for developing useful new abstractions.

4.3 Drag using conservation of energy

Conservation of energy helps analyze drag – one of the most difficult subjects in classical physics. To make drag concrete, try the following home experiment.

4.3.1 Home experiment using falling cones

Photocopy this page and cut out the templates, then tape their edges together to make a cone:



When you drop the small cone and the big cone, which one falls faster? In particular, what is the ratio of their fall speeds $v_{\text{big}}/v_{\text{small}}$? The large cone, having a large area, feels more drag than the small cone does. On the other hand, the large cone has a higher driving force (its weight) than the small cone has. To decide whether the extra weight or the extra drag wins requires finding how drag depends on the parameters of the situation.

However, finding the drag force is a very complicated calculation. The full calculation requires solving the Navier–Stokes equations:

$$(\mathbf{v}\cdot\nabla)\mathbf{v} + \frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{v}.$$

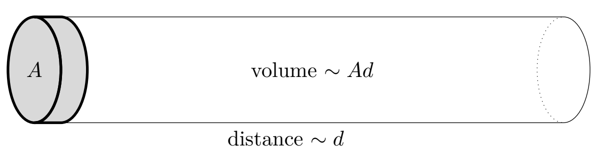
And the difficulty does not end with this set of second-order, coupled, nonlinear partial-differential equations. The full description of the situation includes a fourth equation, the continuity equation:

$$\nabla\cdot\mathbf{v} = 0.$$

One imposes boundary conditions, which include the motion of the object and the requirement that no fluid enters the object – and solves for the pressure p and the velocity gradient at the surface of the object. Integrating the pressure force and the shear force gives the drag force.

In short, solving the equations analytically is difficult. I could spend hundreds of pages describing the mathematics to solve them. Even then, solutions are known only in a few circumstances, for example a sphere or a cylinder moving slowly in a viscous fluid or a sphere moving at any speed in an zero-viscosity fluid. But an inviscid fluid – what Feynman calls 'dry water' [12, Chapter II-40] – is particularly irrelevant to real life since viscosity is the reason for drag, so an inviscid solution predicts zero drag! Conservation of energy, supplemented with skillful lying, is a simple and quick alternative.

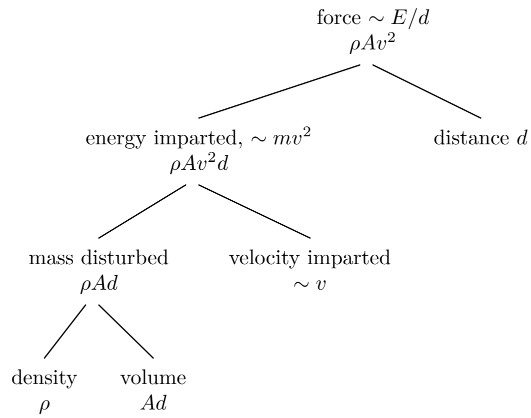
The analysis analysis imagines an object of cross-sectional area A moving through a fluid at speed v for a distance d:



The drag force is the energy consumed per distance. The energy is consumed by imparting kinetic energy to the fluid, which viscosity eventually removes from the fluid. The kinetic energy is mass times velocity squared. The mass disturbed is ρAd, where ρ is the fluid density (here, the air density). The velocity imparted to the fluid is roughly the velocity of the disturbance, which is v. So the kinetic energy imparted to the fluid is ρAv^2d, making the drag force

$$F \sim \rho A v^2.$$

The analysis has a divide-and-conquer tree:



The result that $F_{\text{drag}} \sim \rho v^2 A$ is enough to predict the result of the cone experiment. The cones reach terminal velocity quickly (see **Problem 8.6**), so the relevant quantity in finding the fall time is the terminal velocity. From the drag-force formula, the terminal velocity is

$$v \sim \sqrt{\frac{F_{\text{drag}}}{\rho A}}.$$

The cross-sectional areas are easy to measure with a ruler, and the ratio between the small- and large-cone terminal velocities is even easier. The experiment is set up to make the drag force easy to measure: Since the cones fall at their respective terminal velocities, the drag force equals the weight. So

$$v \sim \sqrt{\frac{W}{\rho A}}.$$

Each cone's weight is proportional to its cross-sectional area, because they are geometrically similar and made out of the same piece of paper. So the terminal velocity v is independent of the area A: so the small and large cones should fall at the same speed.

To test this prediction, I stood on a table and dropped the two cones. The fall lasted about two seconds, and they landed within 0.1 s of one another. So, the approximate conservation-of-energy analysis gains in plausibility (all the inaccuracies are hidden within the changing drag coefficient).