To solve this resulting temperature distribution, there is no need to solve the heat equation. Since all the edges are held at 120°, the temperature throughout the sheet is 120°.

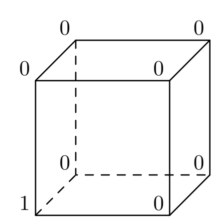
That information is enough to solve the original problem. The symmetry operation is a rotation about the center of the pentagon, so the centers overlap when the plates are stacked atop one another. Since the stacked plate has a temperature of 120° throughout, and the centers of the five stacked sheets align, each center is at $T = 120°/5 = 24°$.

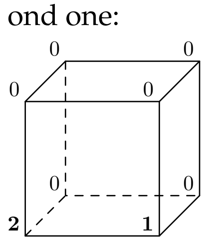
To find transferable ideas, compare the symmetry solutions to Gauss's sum and to the pentagon temperature. Both problems looked complex at first glance. Gauss's sum had many terms in it, all different. The pentagon problem seemed to require solving a difficult differential equation. Both problems contained a symmetry operation. In Gauss's sum, the symmetry operation flipping the sum around. In the pentagon problem, the symmetry operation rotated the pentagon by 72°. In both problems, the symmetry operation left an important quantity unchanged: the sum S or the temperature T_{center}. And this invariance became the key to solving the problem simply.

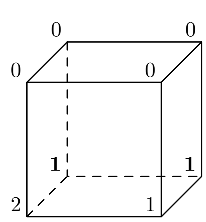
A moral of these two examples is: When there is change, look for what does not change. In other words, look for invariants. Alternatively, if those quantities are given (e.g. the sum S or temperature at the center), look for operations that leave them unchanged. In other words, look for symmetries.

4.2 Cube solitaire

Here is a game of solitaire that illustrates the theme of this chapter. The following cube starts in the configuration in the margin; the goal is to make all vertices be multiples of three simultaneously. The moves are all of the same form: Pick any edge and increment its two vertices by one. For example, if I pick the bottom edge of the front face, then the bottom edge of the back face, the configuration becomes the first one in this series, then the second one:



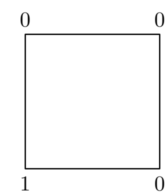




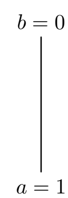
Alas, neither configuration wins the game.

Can I win the cube game? If I can win, what is a sequence of moves ends in all vertices being multiples of 3? If I cannot win, how can that negative result be proved?

Brute force – trying lots of possibilities – looks overwhelming. Each move requires choosing one of 12 edges, so there are 12^{10} sequences of ten moves. That number is an overestimate because the order of the moves does not affect the final state. I could push that line of reasoning by figuring out how many possibilities there are, and how to list and check them if the number is not too large. But that approach is specific to this problem and unlikely to generalize to other problems.



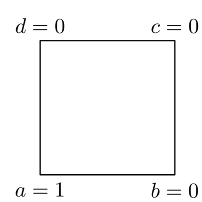
Instead of that specific approach, make the generic observation that this problem is difficult because each move offers many choices. The problem would be simpler with fewer edges: for example, if the cube were a square. Can this square be turned into one where the four vertices are multiples of 3? This problem is not the original problem, but solving it might teach me enough to solve the cube. This hope motivates the following advice: *When the going gets tough, the tough lower their standards.*



The square is easier to analyze than is the cube, but standards can be lowered still more by analyzing the one-dimensional analog, a line. Having only one edge means that there is only one move: incrementing the top and bottom vertices. The vertices start with a difference of one, and continue with that difference. So they cannot be multiples of 3 simultaneously. In symbols: $a - b = 1$. If all vertices were multiples of 3, then $a - b$ would also be a multiple of 3. Since $a - b = 1$, it is also true that

$$a - b \equiv 1 \pmod 3,$$

where the mathematical notation $x \equiv y \pmod 3$ means that x and y have the same remainder (the same modulus) when dividing by 3. In this one-dimensional version of the game, the quantity $a - b$ is an *invariant:* It is unchanged after the only move of increasing each vertex on an edge.

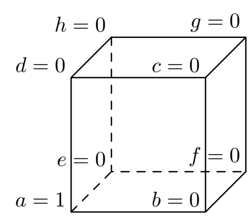


Perhaps a similar invariant exists in the two-dimensional version of the game. Here is the square with variables to track the number at each vertex. The one-dimensional invariant $a - b$ is sometimes an invariant for the square. If my move uses the bottom edge, then a and b increase by 1, so $a - b$ does not change. If my move uses the top edge, then a and b are individually unchanged so $a - b$ is again unchanged. However, if my move uses the left or right edge, then either a or b changes without a compensating change in the other variable. The difference $d - c$ has a similar behavior in that it is changed by some of the moves. Fortunately, even when $a - b$ and $d - c$ change, they change in the same way. A move using the left edge increments $a - b$ and $d - c$; a move using the right edge decrements $a - b$ and $d - c$. So $(a - b) - (d - c)$ is invariant! Therefore for the square,

$$a - b + c - d \equiv 1 \pmod 3,$$

so it is impossible to get all vertices to be multiples of 3.

The original, three-dimensional solitaire game is also likely to be impossible to win. The correct invariant shows this impossibility. The quantity $a-b+c-d+f-g+h-e$ generalizes the invariant for the square, and it is preserved by all 12 moves. So



$$a - b + c - d + f - g + h - e \equiv 1 \pmod 3,$$

which shows that all vertices cannot be made multiples of 3 simultaneously.

Invariants – quantities that remain unchanged – are a powerful tool for solving problems. Physics problems are also solitaire games, and invariants (conserved quantities) are essential in physics. Here is an example: In a frictionless world, design a roller-coaster track so that an unpowered roller coaster, starting from rest, rises above its starting height. Perhaps a clever combination of loops and curves could make it happen.

The rules of the physics game are that the roller coaster's position is determined by Newton's second law of motion $F = ma$, where the forces on the roller coaster are its weight and the contact force from the track. In choosing the shape of the track, you affect the contact force on the roller coaster, and thereby its acceleration, velocity, and position. There are an infinity of possible tracks, and we do not want to analyze each one to find the forces and acceleration. An invariant, energy, simplifies the analysis. No matter what tricks the track does, the kinetic plus potential energy

$$\frac{1}{2}mv^2 + mgh$$

is constant. The roller coaster starts with $v = 0$ and height h_{start}; it can never rise above that height without violating the constancy of the energy. The invariant – the conserved quantity – solves the problem in one step, avoiding an endless analysis of an infinity of possible paths.

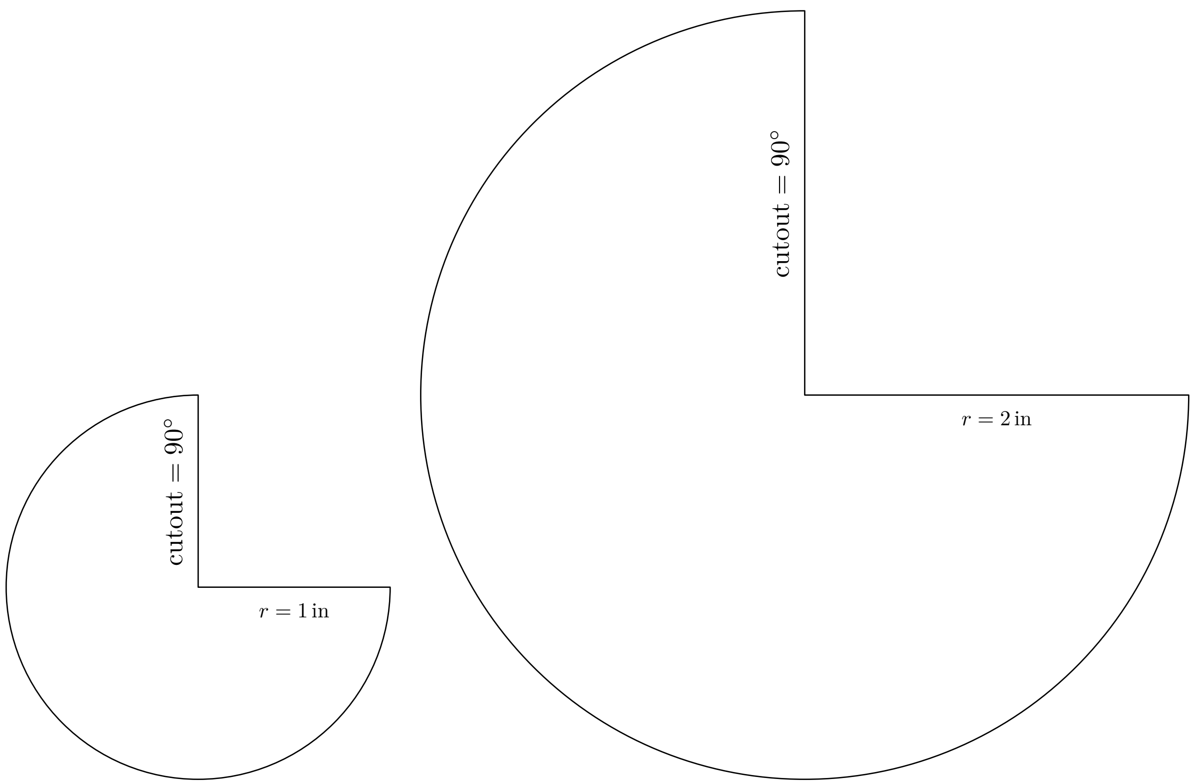
The moral of this section is: *When there is change, look for what does not change.* That quantity becomes a new abstraction (**Chapter 3**), so looking for invariants is a recipe for developing useful new abstractions.

4.3 Drag using conservation of energy

Conservation of energy helps analyze drag – one of the most difficult subjects in classical physics. To make drag concrete, try the following home experiment.

4.3.1 Home experiment using falling cones

Photocopy this page and cut out the templates, then tape their edges together to make a cone:



When you drop the small cone and the big cone, which one falls faster? In particular, what is the ratio of their fall speeds $v_{\mathrm{big}}/v_{\mathrm{small}}$? The large cone, having a large area, feels more drag than the small cone does. On the other hand, the large cone has a higher driving force (its weight) than the