6.055J/2.038J (Spring 2009)

Solution set 05

Do the following warmups and problems. Due in class on ***Wednesday, 13 May 2009****.*

Open universe: *Collaboration, notes, and other sources of information are* ***encouraged.*** *However, avoid looking up answers until you solve the problem (or have tried hard). That policy helps you learn the most from the problems.*

Homework will be graded with a light touch: P (made a reasonable effort), D (did not make a reasonable effort), or F (did not turn in).

Warmups

1. Numerical estimates

Estimate

a. $\sqrt{1.05}$

It is an example of $(1+x)^n$ with $x = 0.05$ and $n = 1/2$. Since x and nx are small, $(1+x)^n \approx 1 + nx$. So

$$\sqrt{1.05} \approx 1 + \frac{1}{2} \times 0.05 = 1.025.$$

The true value is very close: 1.024695 . . ..

b. 1.1^{23}

Here $x = 0.1$ and $n = 23$. So although x is small, $nx = 2.3$ is large (compared to 1). Therefore

$$(1+x)^n \approx e^{nx}.$$

Here, that becomes $e^{2.3}$, which is conveniently 10. The true value is only 10% smaller: 8.95 . . ..

2. Perfume

If the diffusion constant (in air) for small perfume molecules is $10^{-6}\,\mathrm{m^2\,s^{-1}}$, estimate the time for perfume molecules to diffuse across a room.

I'll pick a random but common room, say our classroom. It has size $L \sim 10\,\mathrm{m}$. A diffusion constant D has dimensions L^2T^{-1}, so there's only one way to make a time t out of a length and a diffusion constant:

$$t \sim \frac{L^2}{D} \sim \frac{100\,\mathrm{m}^2}{10^{-6}\,\mathrm{m}^2\,\mathrm{s}^{-1}} = 10^8\,\mathrm{s}.$$

A year is almost exactly $\pi \cdot 10^7$ s, so t is about 3 years. Random walks across long distances are slow!

Now try the experiment: How long does it take to smell the perfume from across the room? Explain the large discrepancy between the theoretical estimate and the experimental value.

I don't often use perfume, but I ran a similar experiment in my apartment by opening a bottle of nail-polish remover. It took a few minutes for the smell to become noticable across the room. The discrepancy is large between the prediction of a few years and the result of a few minutes. So, although the prediction is approximate, the exact calculation (which would include figuring out what concentration produces a noticeable odor) is unlikely to fix such a large discrepancy.

Rather, the physical model on which it is based must not be right. It would be correct if the air were perfectly still. Then the perfume molecules would have no way to get across the room except by diffusion. However, there are always air currents due to wind, convection, people walking, fans, ... Almost no matter how weak those currents are, they are much more efficient at transporting the perfume molecules than diffusion is.

3. Reynolds numbers

Estimate the Reynolds number for:

a. a falling raindrop;

From an earlier problem set, a raindrop falls at about $10\,\mathrm{m\,s^{-1}}$ and it has a radius of roughly 3 mm. So the Reynolds number is

$$\mathrm{Re} = \frac{rv}{\nu} \sim \frac{3 \cdot 10^{-3}\,\mathrm{m} \times 10\,\mathrm{m\,s^{-1}}}{1.5 \cdot 10^{-5}\,\mathrm{m^2\,s^{-1}}} \sim 2000.$$

b. a flying mosquito;

Using reasonable guesses for the flight speed and size:

$$\mathrm{Re} \sim \frac{10^{-3}\,\mathrm{m} \times 1\,\mathrm{m\,s^{-1}}}{10^{-5}\,\mathrm{m^2\,s^{-1}}} \sim 100.$$

Even a mosquito experiences high-Reynolds-number drag. It's not quite turbulent flow, which happens around $\mathrm{Re} \sim 10^3$, but it's still significantly higher than the value for low-Reynolds-number (Stokes) drag.

Problems

4. Drag at low Reynolds number

At low Reynolds number, the drag on a sphere is

$$F = 6\pi\rho\nu v r.$$

What is the drag coefficient c_d as a function of Reynolds number *Re*?

The drag coefficient is

$$c_d \equiv \frac{F}{\frac{1}{2}\rho v^2 A},$$

where $A = \pi r^2$ is the cross-sectional area. So

$$c_d = \frac{6\pi\rho\nu v r}{\frac{1}{2}\rho v^2 \pi r^2} = 12\frac{\nu}{vr} = \frac{12}{\mathrm{Re}}.$$

5. Blackbody temperature of the earth

The earth's surface temperature is mostly due to solar radiation.

The solar flux $S \approx 1350\,\mathrm{W\,m^{-2}}$ is the amount of solar energy reaching the top of the earth's atmosphere. But that energy is spread over the surface of a sphere, so $S/4$ is the relevant flux for calculating the surface temperature. Some of that energy is reflected back to space by clouds or ocean before it can heat the ground, so the heating flux is slightly lower than $S/4$. A useful estimate is $S' \sim 250\,\mathrm{W\,m^{-2}}$.

Look up the Stefan–Boltzmann law (or see **Problem 8**) and use it to find the blackbody temperature of the earth.

Your value should be close to room temperature but enough colder to make you wonder about the discrepancy. Why is the actual average surface temperature warmer than the value calculated in this problem?

According to the Stefan–Boltzmann law, the power per area radiated from a blackbody is $F = \sigma T^4$, where σ is the Stefan–Boltzmann constant and T is the temperature of the body (the object). So

$$T = (F/\sigma)^{1/4}.$$

The Stefan–Boltzmann constant σ is constructed from other fundamental constants (you can derive most of σ using the method of **Problem 8**). It's value is

$$\sigma \approx 5.7 \cdot 10^{-8}\,\mathrm{J\,s^{-1}\,m^{-2}\,K^{-4}}.$$

So,

$$T \approx \left(\frac{250\,\mathrm{W\,m^{-2}}}{5.7 \cdot 10^{-8}\,\mathrm{J\,s^{-1}\,m^{-2}\,K^{-4}}}\right)^{1/4} \sim 257\,\mathrm{K}.$$

In normal units, that's $-16\,^{\circ}\mathrm{C}$ or $3\,^{\circ}\mathrm{C}$. That's very cold, colder than the average Boston winter day.

It is close to room temperature, but the discrepancy is a bit large. What's wrong with the calculation? The greenhouse effect! The earth absorbs the $250\,\mathrm{W\,m^{-2}}$ from the sun, and it radiates it to space. Those parts of the calculation are correct. But the outgoing radiation is mostly infrared, which is well absorbed by carbon dioxide and water molecules in the atmosphere. The absorbed radiation is radiated in all directions, including back to the earth – warming the surface, and making life bearable.

So, we need the greenhouse effect, just not too much of it.

6. Fog

a. Estimate the terminal speed of fog droplets (radius $\sim 10\,\mu\text{m}$). Use either the low- or high-Reynolds-number limit for the drag force, whichever you guess is the more likely to be valid.

Here is the low-Reynolds-number terminal velocity from the lecture notes:

$$v \sim \frac{2}{9}\frac{gr^2}{\nu}\left(\frac{\rho_{\text{obj}}}{\rho_{\text{fl}}} - 1\right).$$

Here ρ_{obj} is the density of water, which is much greater than ρ_{fl}, the density of air. So the -1 is not important. With that simplification and calling $2/9 = 1/4$,

$$v \sim \frac{1}{4} \times \frac{10\,\text{m}\,\text{s}^{-2} \times 10^{-10}\,\text{m}^2}{10^{-5}\,\text{m}^2\,\text{s}^{-1}} \times 1000 \sim 2\,\text{cm}\,\text{s}^{-1}.$$

b. Use the speed to estimate the Reynolds number and check that you used the correct limit for the drag force. If not, try the other limit!

The Reynolds number is roughly

$$\text{Re} \sim \frac{10^{-5}\,\text{m} \times 2\cdot 10^{-2}\,\text{m}\,\text{s}^{-1}}{10^{-5}\,\text{m}^2\,\text{s}^{-1}} \sim 0.02.$$

It is much less than 1, so the original assumption of low-Reynolds-number flow is okay.

c. Fog is a low-lying cloud. How long would fog droplets take to fall 1 km (the height of a typical cloud)? What is the everyday effect of this settling time?

At $2\,\text{cm}\,\text{s}^{-1}$, it takes $5\cdot 10^4$ s to fall 1 km. A day is roughly 10^5 s, so the fall time is about one-half of a day. The everyday consequence is that fog settles overnight: You go to sleep with a pea-soup fog, and by the time you wake up, it's mostly settled onto the ground and maybe evaporated as the morning sun warms the ground.

Optional

7. Plant-watering system

The semester is over and you are going on holiday for a few weeks. But how will you water the house plants?! Design an unpowered slow-flow system to keep your plants happy.

One method is to use Poiseuille flow by choosing the pressure gradient and pipe diameter to get a slow-enough flow.

One of my plants needs a cup of water ($\sim 200\,\text{cm}^3$) every day, so

$$Q \sim \frac{2 \cdot 10^{-4}\,\text{m}^3}{10^5\,\text{s}} \sim 2 \cdot 10^{-9}\,\text{m}^3\,\text{s}^{-1}.$$

From an earlier problem,

$$Q \sim \frac{\Delta p}{l} \frac{r^4}{\rho\nu}.$$

To make Q very tiny, the best way is to use a small pipe radius r, because r shows up with a fourth power. I'll see how well $r = 0.1$ mm works.

Another part of the problem is how to make the pressure gradient. I'll let gravity generate the gradient by keeping the water in a tall tank of height h (with a plastic sheet as a cover to prevent evaporation) and using the hydrostatic pressure $\rho g h$ as the driving pressure. Then $\Delta p = \rho g h$ and

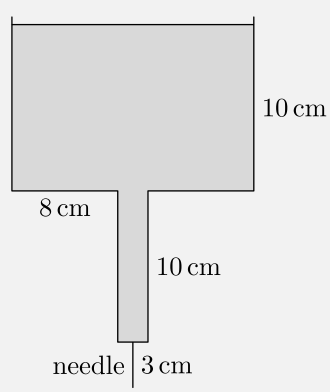
$$Q \sim \frac{gh}{l} \frac{r^4}{\nu} = g \frac{r^4}{\nu} \frac{h}{l}.$$

With $r = 0.1$ mm, the flow rate is

$$Q \sim 10\,\text{m}\,\text{s}^{-2} \times \frac{10^{-16}\,\text{m}^4}{10^{-6}\,\text{m}^2\,\text{s}^{-1}} \times \frac{h}{l} = 10^{-9}\,\text{m}^3\,\text{s}^{-1} \times \frac{h}{l}.$$

So $h/l \sim 2$ in order to get the desired $Q \sim 2 \cdot 10^{-9}\,\text{m}^3\,\text{s}^{-1}$. Actually, if I include the factor of $\pi/8$, then I need $h/l \sim 5$. One way to get $r \sim 0.1$ mm is to use a 27-gauge needle. A typical 27-gauge needle – at least, the ones I've used for giving myself allergy treatments – has $l \sim 3$ cm. So I'll need $h \sim 15$ cm.

It's not easy to keep h fixed at 15 cm for two weeks. But if h does not vary too much, then the flow will be constant enough. I'll let h vary between 10 cm and 20 cm with this arrangement:



The 17 cm width for the big part of the tank allows the tank to contains enough water – roughly 3 liters – to water the plant for a couple weeks.

8. Blackbody radiation

A hot object – a so-called blackbody – radiates energy, and the flux F depends on the temperature T. In this problem you derive the connection using dimensional analysis. The goal is to find F as a function of T. But you need more quantities.

a. What are the dimensions of flux?

b. What two constants of nature should be included because blackbody radiation depends on the quantum theory of radiation?

c. What constant of nature should be included because you are dealing with temperature?

d. After doing the preceding parts, you have five variables. Explain why these five variables produce one dimensionless group, and use that fact to deduce the relation between flux and temperature.

e. Look up the Stefan–Boltzman law and compare your result to it.

9. Teacup spindown

You stir your afternoon tea to mix the milk (and sugar if you have a sweet tooth). Once you remove the stirring spoon, the rotation starts to slow. What is the spindown time τ? In other words, how long before the angular velocity of the tea has fallen by a significant fraction?

To estimate τ, consider a physicist's idea of a teacup: a cylinder with height L and diameter L, filled with liquid. Why does the rotation slow? Tea near the edge of the teacup – and near the base, but for simplicity neglect the effect of the base – is slowed by the presence of the edge (the no-slip boundary condition). The edge produces a velocity gradient.

Because of the tea's viscosity, the velocity gradient produces a force on any piece of the edge. This force tries to spin the piece in the direction of the tea's motion. The piece exerts a force on the tea equal in magnitude and opposite in direction. Therefore, the edge slows the rotation. Now you can analyze this model quantitatively.

a. In terms of the total viscous force F and of the initial angular velocity ω, estimate the spindown time. Hint: Consider torque and angular momentum. (Feel free to drop any constants, such as π and 2, by invoking the Estimation Theorem: $1 = 2$.)

b. You can estimate F with the idea that

$$\text{viscous force} \sim \rho\nu \times \text{velocity gradient} \times \text{surface area}.$$

Here $\rho\nu$ is η. The more familiar viscosity is η, known as the dynamic viscosity. The more convenient viscosity is ν, the kinematic viscosity. The velocity gradient is determined by the size of the region in which the the edge has a significant effect on the flow; this region is called the boundary layer. Let δ be its thickness. Estimate the velocity gradient near the edge in terms of δ, and use the equation for viscous force to estimate F.

c. Put your expression for F into your earlier estimate for τ, which should now contain only one quantity that you have not yet estimated (the boundary-layer thickness).

d. You can estimate δ using your knowledge of random walks. The boundary layer is a result of momentum diffusion; just as D is the molecular-diffusion coefficient, ν is the momentum-diffusion coefficient. In a time t, how far can momentum diffuse? This distance is δ. What is a natural estimate for t? (Hint: After rotating 1 radian, the fluid is moving in a significantly different direction than before, so the momentum fluxes no longer add.) Use that time to estimate δ.

e. Now put it all together: What is the characteristic spindown time τ (the time for the rotation to slow down by a significant amount)?

f. Stir some tea to experimentally estimate τ_{exp}. Compare this time with the time predicted by the preceding theory. [In water (and tea is roughly water), $\nu \sim 10^{-6}\,\mathrm{m}^2\,\mathrm{s}^{-1}$.]

10. Stokes' law

You can use ideas from the previous problem to derive Stokes' formula for drag at low speeds (more precisely, at low Reynolds' number). In lecture we derived the result using dimensional analysis and easy-cases reasoning; in this problem you work out a physical argument.

Consider a sphere of radius R moving with velocity v. Equivalently, in the reference frame of the sphere, the sphere is fixed and the fluid moves past it with velocity v. Next to the sphere, the fluid is stationary. Over a region of thickness δ (the boundary layer), the fluid velocity rises from zero to the full flow speed v. Assume that $\delta \sim R$ (the most natural assumption) and estimate the viscous drag force. Compare the force with Stokes' formula (remember that $\rho\nu = \eta$).