6.055J/2.038J (Spring 2009)

Solution set 4

Do the following warmups and problems. Due in class on ***Monday, 4 May 2009****.*

Open universe: *Collaboration, notes, and other sources of information are* ***encouraged.*** *However, avoid looking up answers until you solve the problem (or have tried hard). That policy helps you learn the most from the problems.*

Homework will be graded with a light touch: P (made a reasonable effort), D (did not make a reasonable effort), or F (did not turn in).

If you have a thesis due on Friday 8 May, you can take an automatic extension until Monday 11 May (there will likely be a final problem set due on 13 May, handed out at least a week before that date).

Warmups

1. Counting dimensionless groups

How many independent dimensionless groups are there in the following sets of variables:

a. size of hydrogen including relativistic effects:

$$e^2/4\pi\epsilon_0,\ \hbar,\ c,\ a_0 \text{ (Bohr radius)},\ m_e \text{ (electron mass)}.$$

According to the Buckingham Pi theorem, five quantities composed of three independent dimensions make two independent dimensionless groups.

The question did not ask you to choose the dimensionless groups. But it is useful to see what they could be. The following pair is a reasonable choice for the two groups:

$$\Pi_1 \equiv \frac{\hbar^2}{m_e a_0 (e^2/4\pi\epsilon_0)} \quad \text{and} \quad \Pi_2 \equiv \frac{e^2/4\pi\epsilon_0}{\hbar c}.$$

The groups are often called Π variables, following a tradition started by Buckingham, author of the Buckingham Pi theorem.

In the preceding choice of groups, the second group Π_2 is the fine-structure constant α.

b. period of a spring–mass system in a gravitational field:

$$T \text{ (period)},\ k \text{ (spring constant)},\ m,\ x_0 \text{ (amplitude)},\ g.$$

Five quantities composed of three independent dimensions make two independent dimensionless groups. Here is a reasonable combination:

$$\Pi_1 \equiv \frac{kT^2}{m} \quad \text{and} \quad \Pi_2 \equiv \frac{kx_0}{mg}.$$

It turns out that the second group Π_2 does not affect the period T. However, dimensional analysis does not tell you that result; it has to be derived by physical thinking.

c. speed at which a free-falling object hits the ground:

v, g, h (initial drop height).

Three quantities composed of two dimensions (length and time) produce one independent dimensionless group. A reasonable choice is

$$\Pi_1 \equiv \frac{v^2}{gh}.$$

That ratio – except for a factor of 2 – has a physical interpretation as the ratio of kinetic energy on impact to the potential energy at the start.

d. [tricky!] weight W of an object:

W, g, m.

These three quantities are composed of three dimensions (mass, length, and time), so there should be zero dimensionless groups! However, there is at least one group: W/mg is dimensionless.

What went wrong is that the three quantities are composed of two *independent* dimensions: mass M and acceleration LT^{-2}. So the Buckingham Pi theorem predicts one independent dimensionless group. A reasonable choice for it is W/mg.

2. Integrals by dimensions

You can use dimensions to do integrals. As an example, try this integral:

$$I(\beta) = \int_{-\infty}^{\infty} e^{-\beta x^2}\, dx.$$

Which choice has correct dimensions:

(a.) $\sqrt{\pi}\beta^{-1}$ **(b.)** $\sqrt{\pi}\beta^{-1/2}$ **(c.)** $\sqrt{\pi}\beta^{1/2}$ **(d.)** $\sqrt{\pi}\beta^{1}$

Hints:

1. The dimensions of dx are the same as the dimensions of x.

2. Pick interesting dimensions for x, such as length. (If x is dimensionless then you cannot use dimensional analysis on the integral.)

An exponent, for example $-\beta x^2$, the question, 'How many times do I multiply the base by itself?' Here the base is e, but the principle applies to any exponent.

Pretend that x is a length. The dimensions of dx are the same as the dimensions of x, so dx is also a length. The exponential is dimensionless since it's the product, $-\beta x^2$ times, of the dimensionless number e. So the integral, which is the sum of little lengths (from the dx), is also a length.

Now determine which choice has dimensions of length. Since x is a length, $[\beta] = L^{-2}$ in order that βx^2 be dimensionless. So $\beta^{-1/2}$ is a length; any other power is something else.

The only reasonable choice is $\sqrt{\pi}\beta^{-1/2}$. I might have been unkind by giving you a choice with the wrong dimensionless constant alongside the correct power of β. But not this time: The factor of $\sqrt{\pi}$ is correct.

Problems

3. How to avoid remembering lots of constants

Many atomic problems, such as the size or binding energy of hydrogen, end up in expressions with $\hbar$, the electron mass m_e, and $e^2/4\pi\epsilon_0$, which is a nicer way to express the squared electron charge. You can avoid having to remember those constants if instead you remember these values instead:

$$\begin{aligned} \hbar c &\approx 200\,\text{eV nm} = 2000\,\text{eV Å} \\ m_e c^2 &\sim 0.5 \cdot 10^6\,\text{eV} \\ \frac{e^2/4\pi\epsilon_0}{\hbar c} &\equiv \alpha \approx \frac{1}{137} \quad \text{(fine-structure constant).} \end{aligned}$$

Use those values to evaluate the Bohr radius in angstroms (1 Å = 0.1 nm):

$$a_0 = \frac{\hbar^2}{m_e(e^2/4\pi\epsilon_0)}.$$

As an example calculation using the $\hbar c$ value, here is the energy of a photon:

$$E = hf = 2\pi\hbar f = 2\pi\hbar\frac{c}{\lambda},$$

where f is its frequency and λ is its wavelength. For green light, $\lambda \sim 600\,\text{nm}$, so

$$E \sim \frac{\overbrace{6}^{2\pi} \times \overbrace{200\,\text{eV nm}}^{\hbar c}}{\underbrace{600\,\text{nm}}_{\lambda}} \sim 2\,\text{eV}.$$

Looking at the Bohr radius

$$a_0 = \frac{\hbar^2}{m_e(e^2/4\pi\epsilon_0)},$$

I see two powers of $\hbar$ just asking to be joined with two powers of c to make $(\hbar c)^2$. To add two powers of c without changing the fraction, multiply the denominator by c^2 as well. The electron mass m_e is happy to join with two powers of c to make $m_e c^2$, the rest energy of the electron.

Therefore,

$$a_0 = \frac{\hbar^2 c^2}{m_e c^2 (e^2/4\pi\epsilon_0)},$$

Now use

$$\alpha = \frac{e^2/4\pi\epsilon_0}{\hbar c}$$

to simplify even more:

$$a_0 = \frac{\hbar c}{m_e c^2} \underbrace{\frac{\hbar c}{e^2/4\pi\epsilon_0}}_{\alpha^{-1}} = \alpha^{-1} \frac{\hbar c}{m_e c^2}.$$

The pieces of this form are easy to remember:

$$a_0 = \alpha^{-1} \frac{\hbar c}{m_e c^2} \sim 10^2 \times \frac{2000\,\text{eV}\,\text{Å}}{0.5 \cdot 10^6\,\text{eV}} \sim 0.5\text{Å}.$$

4. Heavy nuclei

In lecture we analyzed hydrogen, which is one electron bound to one proton. In this problem you study the innermost electron in an atom such as uranium that has many protons, and analyze one physical consequence of its binding energy.

So, imagine a nucleus with Z protons around which orbits one electron. Let $E(Z)$ be the binding energy (the hydrogen energy is the case $Z = 1$).

a. Show that the ratio $E(Z)/E(1)$ is Z^2.

With Z protons pulling on one electron, the electrostatic energy is $Ze^2/4\pi\epsilon_0$. So instead of using $e^2/4\pi\epsilon_0$ as one of the quantities, the dimensional analysis should use $Ze^2/4\pi\epsilon_0$. The other quantities are unchanged. The Z propagates along with the e^2 through the calculation of the radius a_Z and the energy $E(Z)$.

Since the radius a_0 has one factor of e^2 in the denominator, the a_Z picks up a factor of Z in the denominator relative to a_0:

$$a_Z = \frac{a_0}{Z}.$$

Since the energy E_0 has a factor of e^4 in the numerator, the energy $E(Z)$ picks up a factor of Z^2:

$$E(Z) = E(1) \times Z^2.$$

b. In lecture, we derived that $E(1)$ is the kinetic energy of an electron moving with speed αc where α is the fine-structure constant (roughly 10^{-2}). How fast does the innermost electron move around a heavy nucleus with charge Z?

To increase the energy of the electron by a factor of Z^2, the speed must increase by a factor of Z. So the innermost electron moves with speed $Z\alpha c$.

c. When that speed is comparable to the speed of light, the electron has a kinetic energy comparable to its (relativistic) rest energy. One consequence of such a high kinetic energy is that the electron has enough kinetic energy to produce a positron (an anti-electron) out of nowhere ('pair creation'). That positron leaves the nucleus, turning a proton into a neutron as it exits. So the atomic number Z drops by one: The nucleus is unstable! Relativity sets an upper limit for Z.

Estimate that maximum Z and compare it with the Z for the heaviest stable nucleus (uranium).

> When $Z\alpha c \sim c$, the electron becomes significantly relativistic and permits pair creation to destabilize the nucleus. So the maximum Z is roughly α^{-1} or about 137.
>
> The heaviest stable nucleus is uranium with $Z = 92$, so the estimate is not too bad.

5. Power radiated by an accelerating charge

Electromagnetism, where the usual derivations are so cumbersome, is an excellent area to apply dimensional analysis. In this problem you work out the power radiated by an accelerating charge, which is how radio stations work.

So, consider a particle with charge q, with position x, velocity v, and acceleration a. What variables are relevant to the radiated power P? The position cannot matter because it depends on the origin of the coordinate system, whereas the power radiated cannot depend on the origin. The velocity cannot matter because of relativity: You can transform to a reference frame where $v = 0$, but that change will not affect the radiation (otherwise you could distinguish a moving frame from a non-moving frame, in violation of the principle of relativity). So the acceleration a is all that's left to determine the radiated power. [This line of argument is slightly dodgy, but it works for low speeds.]

a. Using P, $q^2/4\pi\epsilon_0$, and a, how many dimensionless quantities can you form?

> Zero! There are three quantities and three independent dimensions, so the Buckingham Pi theorem says there will be no dimensionless groups.

b. Fix the problem in the previous part by adding one quantity to the list of variables, and give a physical reason for including the quantity.

> Radiation travels at the speed of light. Furthermore, radiation is a relativistic phenomenon. Relativity means that magnetic fields are electric fields seen from a moving reference frame. And radiation is an alternation of electric and magnetic fields, like a cat chasing its tail, so radiation needs relativity. And relativity needs the speed of light.
>
> Therefore, I should add c to the list of variables.

c. With the new list, use dimensionless groups to find the power radiated by an accelerating point charge. In case you are curious, the exact result contains a dimensionless factor of 2/3; dimensional analysis triumphs again!

Four quantities composed of three dimensions produce one independent group. The dimensions of power P are ML^2T^{-3}. The dimensions of $q^2/4\pi\epsilon_0$ are energy times distance, which is ML^3T^{-2}. The dimensions of acceleration a are LT^{-2}. And the dimensions of c are LT^{-1}.

In order to cancel the mass dimensions, a dimensionless combination will contain $P/(q^2/4\pi\epsilon_0)$, which has dimensions of $L^{-1}T^{-1}$. The only way to make those dimensions from a and c is a^2/c^3. So a dimensionless group is

$$\Pi_1 \equiv \frac{P/(q^2/4\pi\epsilon_0)}{a^2/c^3} = \frac{Pc^3}{a^2(q^2/4\pi\epsilon_0)}.$$

The radiated power P must therefore be

$$P \sim \frac{a^2q^2/4\pi\epsilon_0}{c^3}.$$

This result explains why radiation (radio waves, light, etc.) is such a good way to transmit information for a long distance. To see how, I'll use the power to estimate the electric field at a distance r from the accelerating (and radiating) charge.

The power P is the same no matter how far away one is from the charge because all the radiated energy escapes to infinity. Pretend that the angular distribution of the power is isotropic. Then the radiated power *per area* at a distance r is

$$\mathcal{P} \sim \frac{P}{\text{area of sphere with radius } r} \sim \frac{P}{r^2}.$$

The power per area is also known as the energy flux, which is energy per area per time. Energy flux is the propagation speed times the energy per volume (the energy density) $\mathcal{E}$ in the electric field:

$$\mathcal{P} \propto c\mathcal{E}.$$

Hard to believe? Check the dimensions!

The energy density in the electric field is proportional to the square of the field E:

$$\mathcal{E} \propto E^2.$$

This relation is analogous to the relation that the energy in a spring is proportional to the square of the amplitude.

Put these relations together to find the relation between energy flux and electric field:

$$\mathcal{P} \propto cE^2.$$

Since $\mathcal{P} \sim P/r^2$:

$$E \propto r^{-1}.$$

Compare this distance dependence with the similar result for electrostatics: There the electric field is proportional to r^{-2}. So, by accelerating a charge, we make an electric field that falls off much more slowly than it would from just electrostatics. The importance of that change grows as r grows. For large r – from radio stations and, especially, from stars – radiation is the only way to get a message to us. What a difference that change in the exponent makes!

6. Yield from an atomic bomb

Geoffrey Taylor, a famous Cambridge fluid mechanic, annoyed the US government by doing the following analysis. The question he answered: 'What was the yield (in kilotons of TNT) of the first atomic blast (in the New Mexico desert in 1945)?' Declassified pictures, which even had a scale bar, gave the following data on the radius of the explosion at various times:

t (ms)	R (m)
3.26	59.0
4.61	67.3
15.0	106.5
62.0	185.0

a. Use dimensional analysis to work out the relation between radius R, time t, blast energy E, and air density ρ.

Four quantities composed of three independent dimensions make one independent dimensionless group. The only quantities whose dimensions contain mass are E and ρ. So E and ρ appear in the group as E/ρ, whose dimensions are L^5T^{-2}. Therefore the following choice is dimensionless:

$$\Pi_1 \equiv \frac{Et^2}{\rho R^5}.$$

b. Use the data in the table to estimate the blast energy E (in Joules).

With only one dimensionless group, the most general statement connecting those four quantities is

$$\frac{Et^2}{\rho R^5} \sim 1.$$

or

$$E \sim \frac{\rho R^5}{t^2}.$$

For each row of data in the table, I'll estimate $\rho R^5/t^2$, using $\rho \sim 1\,\mathrm{kg\,m^{-3}}$:

t (ms)	R (m)	E (10^{13} J)
3.26	59.0	6.7
4.61	67.3	6.5
15.0	106.5	6.1
62.0	185.0	5.6

The data are not perfectly consistent about the predicted blast energy E, but they hover pretty closely around $6 \cdot 10^{13}$ J.

c. Convert that energy to kilotons of TNT. One gram of TNT releases 1 kcal or roughly 4 kJ.

One kiloton of TNT is

$$1\,\text{kiloton} \times \frac{10^9\,[\text{g}]}{1\,\text{kiloton}} \times \frac{4 \cdot 10^3\,\text{J}}{1\,[\text{g}]} \sim 4 \cdot 10^{12}\,\text{J}.$$

The predicted yield of $6{\cdot}10^{13}$ J is roughly 15 kiltons, in close agreement with the classified value of 20 kiltons.

If I'd used the more accurate density $\rho \sim 1.3\,\text{kg}\,\text{m}^{-3}$, then I'd have found $E \sim 19.5$ kilotons, a result that is far too accurate for the number of approximations contained in it!

The actual value was 20 kilotons, a classified number when Taylor published his result ['The Formation of a Blast Wave by a Very Intense Explosion. II. The Atomic Explosion of 1945.', *Proceedings of the Royal Society of London. Series A, Mathematical and Physical* **201**(1065): 175–186 (22 March 1950)]

7. **Your turn to create**
Invent – but you do not need to solve! – an estimation question that dimensional analysis would help solve.

Optional

8. **Atomic blast: A physical interpretation**
Use energy densities and sound speeds to make a rough physical explanation of the result in the 'yield from an atomic bomb' problem.

I'll use the blast energy E, the radius R, and the air density ρ to estimate out how fast the fireball expands when it has radius R.

For a simple model, imagine that the fireball expands as fast as the speed of sound. Of course, the speed of sound is much higher in the fireball than in normal air, because the fireball has a huge pressure due to the blast.

The energy density $\mathcal{E}$ in the fireball is the blast energy E divided by the volume of the fireball, so $\mathcal{E} \sim E/R^3$. The energy density is the pressure, give or take a dimensionless constant. Sound speed depends on pressure and density (check the dimensions):

$$c_s \sim \sqrt{\frac{\text{pressure}}{\text{density}}}.$$

So the sound speed, which is the rate at which the fireball expands, is

$$c_s \sim \sqrt{\frac{E}{\rho R^3}}.$$

Next assume that the expansion speed is constant, even though it falls as fireball grows. Then the radius is $R \sim tc_s$. So

$$R \sim t\sqrt{\frac{E}{\rho R^3}},$$

or

$$E \sim \frac{\rho R^5}{t^2},$$

as derived using dimensions.