6.055J/2.038J (Spring 2009)

Solution set 3

Do the following warmups and problems. Due in class on ***Friday, 03 Apr 2009****.*

Open universe: *Collaboration, notes, and other sources of information are* ***encouraged.*** *However, avoid looking up answers until you solve the problem (or have tried hard). That policy helps you learn the most from the problems.*

Homework will be graded with a light touch: P (made a reasonable effort), D (did not make a reasonable effort), or F (did not turn in).

Warmups

1. Minimum power

In lecture we estimated the flight speed that minimizes energy consumption. Call that speed v_E. We could also have estimated v_P, the speed that minimizes power consumption. What is the ratio v_P/v_E?

The zillions of constants (such as ρ) clutter the analysis without changing the result. So I'll simplify the problem by using a system of units where all the constants are 1. Then the energy is

$$E \sim v^2 + \frac{1}{v^2},$$

where the first term is from drag and the second term is from lift. The power is energy per time, and time is inversely proportional to v, so $P \propto Ev$ and

$$P \sim v^3 + \frac{1}{v}.$$

The first term is the steep v^3 dependence of drag power on velocity (which we used to estimate the world-record cycling and swimming speeds).

The energy expression is unchanged when $v \to 1/v$, so it has a minimum at $v_E = 1$. To minimize the power, use calculus (ask me if you are curious about calculus-free ways to minimize it):

$$\frac{dP}{dv} \sim 3v^2 - \frac{1}{v^2} = 0,$$

therefore $v_P = 3^{-1/4}$ (roughly 3/4), which is also the ratio v_P/v_E.

So the minimum-power speed is about 25 less than the minimum-energy speed. That result makes sense. Drag power grows very fast as v increases – much faster than lift power decreases – so it's worth reducing the speed a little to reduce the drag a lot.

If you don't believe the simplification that I used of setting all constants to 1 – and it is not immediately obvious that it should work – then try using this general form:

$$E \sim Av^2 + \frac{B}{v^2},$$

where A and B are constants. You'll find that v_E and v_P get the same function of A and B, which disappears from the ratio v_P/v_E.

2. Solitaire

You start with the numbers 3, 4, and 5. At each move, you choose any two of the three numbers – call the choices a and b – and replace them with $0.8a - 0.6b$ and $0.6a + 0.8b$. The goal is to reach 4, 4, 4. Can you do it? If yes, give a move sequence; if no, show that you cannot.

To see whether solitaire games are solvable, look for an invariant. Alas there is no algorithm for finding invariants; you have to use clues and make lucky guesses.

Speaking of clues, is it a happy coincidence that $0.8^2 + 0.6^2 = 1$? That convenient sum suggests looking at sums of squares, and how those are changed by making a move. Replacing a and b by $a' = 0.8a - 0.6b$ and $b' = 0.6a + 0.8b$ makes the sum of squares $a^2 + b^2$ into $a'^2 + b'^2$. Expand that expression:

$$\begin{aligned} a'^2 + b'^2 &= (0.8a - 0.6b)^2 + (0.6a + 0.8b)^2 \\ &= 0.64a^2 - 0.96ab + 0.36b^2 \; + \; 0.36a^2 + 0.96ab + 0.64b^2 \\ &= a^2 + b^2. \end{aligned}$$

Great! Each move leaves the sum of squares unchanged. That sum started out with the invariant at $3^2 + 4^2 + 5^2 = 50$, so it remains 50. The goal state, however, requires that the invariant become $4^2 + 4^2 + 4^2 = 48$. It's not possible to reach the goal.

The invariant has a nice geometric interpretation (a picture). To see it, let $P = (a, b, c)$ be the coordinates of a point in three-dimensional space. Then each move leaves unchanged the distance to the origin, which is $\sqrt{a^2 + b^2 + c^2}$. So each move shifts P to another location equally distant from the origin, meaning that it moves P on the surface of a sphere. But it cannot escape the surface.

An interesting question to which I don't know the answer: Can you reach every point on the surface of the sphere? The distance invariant does not forbid it, but maybe other constraints do?

3. Highway vs city driving

Here is a measure of the importance of drag for a car moving at speed v for a distance d:

$$\frac{E_{drag}}{E_{kinetic}} \sim \frac{\rho v^2 A d}{m_{car} v^2}.$$

a. Show that the ratio is equivalent to the ratio

$$\frac{\text{mass of the air displaced}}{\text{mass of the car}}$$

and to the ratio

$$\frac{\rho_{air}}{\rho_{car}} \times \frac{d}{l_{car}},$$

where ρ_{car} is the density of the car (i.e. its mass divided by its volume) and l_{car} is the length of the car.

In the ratio $\rho v^2 A d / m_{car} v^2$, the v^2 divide out leaving $\rho A d / m_{car}$, where ρ is the air density. Since A is the cross-sectional area of the car, Ad is the volume of air that the car displaces, and $\rho A d$ is the mass of that air. So

$$\frac{E_{drag}}{E_{kinetic}} \sim \frac{\rho v^2 A d}{m_{car} v^2} = \frac{\rho A d}{m_{car}} = \frac{\text{mass of the air displaced}}{\text{mass of the car}}.$$

An alternative equivalence comes from writing the mass of the car as $\rho_{car} A l_{car}$. Making that substitution and dividing out the v^2 gives

$$\frac{\rho v^2 A d}{m_{car} v^2} = \frac{\rho_{air} A d}{\rho_{car} A l_{car}} = \frac{\rho_{air}}{\rho_{car}} \frac{d}{l_{car}}.$$

b. Make estimates for a typical car and find the distance d at which the ratio becomes significant (say, roughly 1). How does the distance compare with the distance between exits on the highway and between stop signs or stoplights on city streets?

A typical car has mass $m_{car} \sim 10^3$ kg, cross-sectional area $A \sim 2\,\text{m} \times 1.5\,\text{m} = 3\,\text{m}^2$, and length $l_{car} \sim 4$ m. So

$$\rho_{car} \sim \frac{m_{car}}{A l_{car}} \sim \frac{10^3\,\text{kg}}{3\,\text{m}^2 \times 4\,\text{m}} \sim 10^2\,\text{kg}\,\text{m}^{-3}.$$

Since $\rho_{car}/\rho_{air} \sim 100$, the ratio

$$\frac{\rho_{air}}{\rho_{car}} \frac{d}{l_{car}}$$

becomes 1 when $d/l_{car} \sim 100$, so $d \sim 400$ m.

This distance d is significantly farther than the distance between stop signs or stoplights on city streets. In Manhattan, for example, 20 east–west blocks are one mile, giving a spacing of approximately 80 m. So air resistance is not a significant loss in city driving. Instead the loss comes from engine friction, rolling resistance, and braking.

However, the distance d is comparable to the exit spacing on urban highways. So when you drive on the highway for even a few exit distances, air resistance is a significant loss.

Interestingly, highway fuel efficiencies are higher than city fuel efficiencies, even though drag gets worse at the higher, highway speeds, and presumably engine friction and rolling resistance also get worse at higher speeds. Only one loss mechanism, braking, is less prevalent in highway than in city driving. So braking must cause a significant loss in city driving. Regenerative braking, for hybrid or electric cars, should significantly improve fuel efficiency in city driving.

4. Mountains

Look up the height of the tallest mountain on earth, Mars, and Venus, and explain any pattern in the three heights.

The heights are:

– Mars: 27 km (Mount Olympus)

- earth: 9 km (Mount Everest)
- Venus: 11 km (Maxwell Montes)

One pattern is that the large planets (earth and Venus) have short mountains, at least short compared to Mount Olympus at a huge height of 27 km.

Large planets presumably have stronger gravitational fields at their surface, which keeps the mountains closer to the ground. The derivation in lecture on mountain heights dropped the dependence on g because we looked only at mountains on earth. Here's the same derivation but retaining g. The weight of a mountain of size l is $W \propto gl^3$, so the pressure at the base is $p \propto gl^3/l^2 \sim gl$. When the pressure exceeds the maximum pressure that rock can support, the mountain can no longer grow upward. So the maximum height l depends inversely on g:

$$l \propto g^{-1}.$$

To test that analysis, here are the gravitational field strengths on the three planets:

- Mars: $3.7\,\text{m}\,\text{s}^{-2}$
- earth: $10\,\text{m}\,\text{s}^{-2}$
- Venus: $8.9\,\text{m}\,\text{s}^{-2}$

The product gl for each planet should be the same, and it roughly is:

- Mars: $10^5\,\text{m}^2\,\text{s}^{-2}$
- earth: $0.9 \cdot 10^5\,\text{m}^2\,\text{s}^{-2}$
- Venus: $0.98 \cdot 10^5\,\text{m}^2\,\text{s}^{-2}$

Fun question: Why aren't mountains on the moon 60 km tall (since the Moon's surface gravity is about one-sixth of earth's surface gravity)?

Problems

5. Raindrop speed

a. How does a raindrop's terminal velocity v depend on the raindrop's radius r?

The weight of the raindrop is the density times the volume times g:

$$W \sim \rho r^3 g,$$

where I neglect dimensionless factors such as $4\pi/3$.

At terminal velocity, the weight equals the drag. The drag is

$$F \sim \rho_{air} v^2 A \sim \rho_{air} v^2 r^2.$$

Equating the weight to the drag gives an equation for v and r:

$$\rho_{air} v^2 r^2 \sim \rho r^3 g,$$

so $v \propto r^{1/2}$.

Bigger raindrops fall faster but – because of the square root – not much faster.

b. Estimate the terminal speed for a typical raindrop.

With the g and the densities, the terminal velocity is

$$v \sim \sqrt{\frac{\rho}{\rho_{air}} gr}.$$

A typical raindrop has a diameter of maybe 6 mm, so $r \sim 3\,\text{mm}$. Since the density ratio between water and air is roughly 1000,

$$v \sim \sqrt{1000 \times 10\,\text{m}\,\text{s}^{-2} \times 3 \cdot 10^{-3}\,\text{m}} \sim 5\,\text{m}\,\text{s}^{-1}.$$

c. How could you check your estimate in part (b)?

First convert the speed into a more familiar value: 11 mph (miles per hour). If one drives at a speed v_{car}, then raindrops appear to move at an angle $\arctan(v_{car}/v)$. When $v_{car} = v$, the drops come at a 45-degree angle. So one way to measure the terminal speed is to drive in a rainstorm, slowly accelerating while the passenger says when the drops come at a 45-degree angle.

You could also run in a rainstorm and note the speed at which a small umbrella has to be at 45 degrees to keep you perfectly dry.

6. Bird flight

a. For geometrically similar animals (same shape and composition but different size), how does the minimum-energy speed v depend on mass M and air density ρ? In other words, what are the exponents α and β in $v \propto \rho^\alpha M^\beta$?

From the lecture notes,

$$Mg \sim C^{1/2} \rho v^2 L^2,$$

where C is the modified drag coefficient. So

$$v \sim \left(\frac{Mg}{C^{1/2}\rho L^2}\right)^{1/2}.$$

For geometrically similar animals, g is independent of size (they all fight the same gravity) and C is also independent of size (because the drag coefficient depends only on shape). But M depends on L according to $M \propto L^3$ or $L \propto M^{1/3}$. So the L^2 in the denominator is proportional to $M^{2/3}$ making

$$v \propto \rho^{-1/2} M^{1/6}.$$

giving $\alpha = -1/2$ and $\beta = 1/6$.

The inverse relationship between the speed and density explains why planes fly at a high altitude. The energy consumption at the minimum-energy speed is independent of ρ, so by flying high where ρ is low, planes increase their speed without increasing their energy consumption.

b. Use that result to write the ratio $v_{747}/v_{\text{godwit}}$ as a product of dimensionless factors, where v_{747} is the minimum-energy speed of a 747, and v_{godwit} is the minimum-energy speed of a bar-tailed godwit. Then estimate the dimensionless factors and their product. Useful information: $m_{\text{godwit}} \sim 0.4\,\text{kg}$.

Assuming that the animals and planes fly at the minimum-energy speed,

$$\frac{v_{747}}{v_{\text{godwit}}} = \left(\frac{\rho_{\text{high}}}{\rho_{\text{sealevel}}}\right)^{-1/2} \cdot \left(\frac{m_{747}}{m_{\text{godwit}}}\right)^{1/6}.$$

A plane flies at around 10 km where the density is roughly one-third of the sea-level density. The mass of a 747 is roughly $4\cdot10^5$ kg, so the mass ratio is 10^6. Therefore the speed ratio should be roughly

$$(1/3)^{-1/2} \times (10^6)^{1/6} = \sqrt{3} \times 10 \sim 17.$$

c. Use v_{747}, from experience or from looking it up, to find v_{godwit}. Compare with the speed of the record-setting bar-tailed godwit, which made its 11,570 km journey in 8 days, 12 hours.

A 747 flies at around 600 mph so the godwit should fly around 600/17 mph $\sim$ 35 mph. The speed of record-setting godwit is

$$\frac{11,570\,\text{km}}{8.5\,\text{days}} \times \frac{0.6\,\text{mi}}{1\,\text{km}} \times \frac{1\,\text{day}}{24\,\text{hours}} \sim 35\,\text{mph}.$$

That's absurdly close to the prediction.

7. Checking plane fuel-efficiency calculation

This problem offers two more methods to estimate the fuel efficiency of a plane.

a. Use the cost of a plane ticket to estimate the fuel efficiency of a 747, in passenger–miles per gallon.

A roundtrip ticket from New York to San Francisco costs roughly $400. The journey is about 2500 miles each way, so a 5000-mile journey costs about $500 (rounding up the $400 to make the math easier). That's about 10 cents/mile. Perhaps one-half of that cost is fuel. [Although the service – in the air, on the phone, and at the counter – is so lousy due to understaffing that perhaps two-thirds of the cost being fuel would be a better estimate!] At 5 cents/mile for fuel, and at $3/gallon for fuel, the fuel efficiency is 60 passenger–miles per gallon.

b. According to Wikipedia, a 747-400 can hold up to $2 \cdot 10^5\,\ell$ of fuel for a maximum range of $1.3 \cdot 10^4$ km. Use that information to estimate the fuel efficiency of the 747, in passenger–miles per gallon.

The 747 can hold about 400 people, so the fuel efficiency is

$$\frac{400\text{ passengers} \times 1.3 \cdot 10^4\text{ km}}{2 \cdot 10^5\,\ell} \times \frac{1\text{ mile}}{1.6\text{ km}} \times \frac{4\ell}{1\text{ gallon}} \sim 65\text{ passenger–miles per gallon.}$$

This estimate is amazingly close to the estimate from using the ticket price!

How do these values compare with the rough result from lecture, that the fuel efficiency is comparable to the fuel efficiency of a car?

The fuel efficiency of a medium-sized car (holding one person, which is typical in much commute traffic) is roughly 30 passenger–miles per gallon. So both fuel-efficiency estimates in this problem give a fuel efficiency that is a factor of 2 higher than the result from lecture – not too bad considering how much we neglected (drag coefficient and lift being the main ones) when we estimated the efficiency.

Optional

8. Inertia tensor

[For those who know about inertia tensors.] Here is the inertia tensor (the generalization of moment of inertia) of a particular object, calculated in a lousy coordinate system:

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 5 & 4 \\ 0 & 4 & 5 \end{pmatrix}$$

a. Change coordinate systems to a set of principal axes. In other words, write the inertia tensor as

$$\begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}$$

and give I_{xx}, I_{yy}, and I_{zz}. *Hint:* What properties of a matrix are invariant when changing coordinate systems?

Whatever coordinate change I make, I will leave the x axis alone because the I_{xx} component is already separated from the y- and z submatrix. That submatrix is

$$\begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$$

I have to figure out how changing the coordinate system changes this submatrix. Rather than find the coordinate change explicitly, I use invariants to avoid that computation.

One invariant of any matrix, not just of this 2×2 matrix, is its determinant. Another invariant is its trace (the sum of the diagonal elements). In the nasty coordinate system, the trace of the y- and z submatrix is $5+5 = 10$. So the trace is 10 in the nice coordinate system. The determinant is $5 \times 5 - 4 \times 4 = 9$, so it the determinant is 9 in the nice coordinate system.

Those facts are sufficient to deduce the submatrix in the nice coordinate system (without needing to figure out what the nice coordinate system is). In the nice coordinate system, the 2×2 submatrix looks like

$$\begin{pmatrix} I_{yy} & 0 \\ 0 & I_{zz} \end{pmatrix}$$

So I need to find I_{yy} and I_{zz} such that

$$I_{yy} + I_{zz} = 10 \quad \text{(from the trace invariant)}$$

and

$$I_{yy} I_{zz} = 9 \quad \text{(from the determinant invariant)}$$

The solution is $I_{yy} = 1$ and $I_{zz} = 9$ (or vice versa). So the inertia tensor becomes

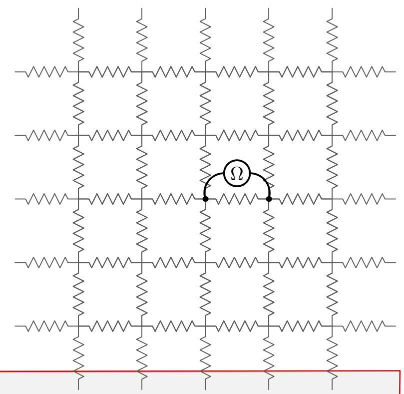
$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

b. Give an example of an object with a similar inertia tensor. On Friday in class we'll have a demonstration.

The object has three principal axes, each with a different moment of inertia. If the object is rectangular and uniform density, the three axes must have different lengths. Most books fit into this category. They have a short axis that passes perpendicularly through the pages (this axis is the one with the highest moment of inertia). The medium-length axis is perpendicular to the spine. And the long axis is parallel to the spine.

9. Resistive grid

In an infinite grid of 1-ohm resistors, what is the resistance measured across one resistor?



To measure resistance, an ohmmeter injects a current I at one terminal (for simplicity, say $I = 1\,\mathrm{A}$), removes the same current from the other terminal, and measures the resulting voltage difference V between the terminals. The resistance is $R = V/I$.

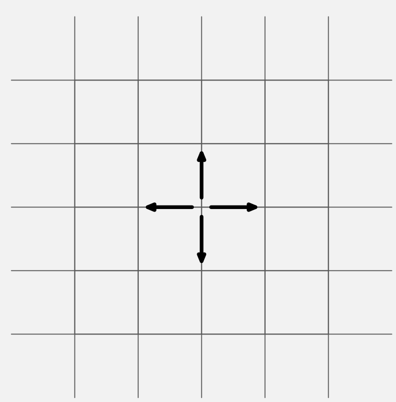
Hint: Use symmetry. But it's still a hard problem!

I'd like to find the current flowing through the resistor when 1 A is sent into one terminal of the ohmmeter and removed from its other terminal. The solution has two steps, each subtle:

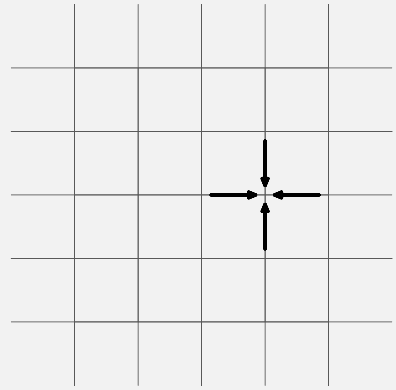
1. Break the resistance-measuring experiment into two parts, each having a lot of symmetry.
2. Analyze those parts using symmetry.

The current distribution that results from the full resistance-measuring experiment is not sufficiently symmetric because it has a preferred direction along the selected resistor. However, if I break the experiment into two parts – inserting current and removing current – then each part produces a symmetric current distribution.

By symmetry – because all four coordinate directions are equivalent – inserting 1 A produces 1/4 A flowing in each coordinate direction away from the terminal. Let's call this terminal the positive terminal. So inserting the 1 A at the positive terminal produces 1/4 A through the selected resistor, and this current flows away from the positive terminal.



By symmetry, removing 1 A produces 1/4 A in each coordinate direction, flowing toward the terminal. Let's call this terminal the negative terminal. So removing 1 A produces 1/4 A through the selected resistor, flowing toward the negative terminal. Equivalently, it produces 1/4 A flowing away from the positive terminal.



Now superimpose the two pictures to reproduce the experiment of measuring the resistance. The experiment produces 1/2 A through the resistor, flowing from the positive to the negative terminal. The voltage across the resistor is the current times its resistance, so the voltage is 1/2 V. Since a 1 A test current produces a 1/2 V drop, the effective resistance is 1/2 Ω.

If you want an even more difficult problem: Find the resistance measured across a diagonal!