

# 6.055J/2.038J (Spring 2009)

## Solution set 1

Do the following warmups and problems. Due in class on **Tuesday, 17 Feb 2009**.

**Open universe:** Collaboration, notes, and other sources of information are **encouraged**. However, avoid looking up answers until you solve the problem (or have tried hard). That policy helps you learn the most from the problems.

Homework will be graded with a light touch: P (made a reasonable effort), D (did not make a reasonable effort), or F (did not turn in).

### Warmups

#### 1. One or few

Use the 1 or 'few' method of multiplication (and division) to do the following calculations mentally, and compare the approximate result with the actual answer:

- a. A random multiplication problem generated by a short Python program:

$$161 \times 294 \times 280 \times 438.$$

The actual product is  $\approx 5.8 \cdot 10^9$ .

The first step is to convert each factor in the product to the nearest power of ten, perhaps also including a factor of a few. For example, 161 contains two factors of 10 and a factor of 1.61; and 1.61 is closer, on a log scale, to 1 than it is to few ( $\sqrt{10}$ ). So 161 becomes simply 100 or  $10^2$ . Here are the conversions for all four factors:

$$161 \rightarrow 10^2$$

$$294 \rightarrow 10^2 \times \text{few};$$

$$280 \rightarrow 10^2 \times \text{few};$$

$$438 \rightarrow 10^2 \times \text{few}.$$

Now the product is easy to do mentally. There are eight factors of 10 and three factors of a few. Since  $(\text{few})^2 = 10$ , three factors of a few becomes  $10 \times \text{few}$ . So

$$161 \times 294 \times 280 \times 438 \approx 10^8 \times 10 \times \text{few} \approx 3 \cdot 10^9,$$

only a factor of 2 smaller than the actual value.

- b. Earth's surface area  $A = 4\pi R^2$ , where the radius is  $R \sim 6 \cdot 10^6$  m. The actual surface area is  $5.1 \cdot 10^{14}$  m<sup>2</sup>.

The surface area is

$$A = 4 \times \pi \times \underbrace{6 \cdot 10^6 \text{ m} \times 6 \cdot 10^6 \text{ m}}_{R^2}.$$

First pull out the factors of 10, of which there are 12, leaving the product  $4 \times \pi \times 6 \times 6$ . Now convert each of these factors using 1 or few. The first two, 4 and  $\pi$ , each become a few. However, a factor of 6 is almost as close to few (roughly 3.2) as it is to 10 – being roughly a factor of 2 from either limit. So, should 6 be converted into a few or into 10?

If there were only one factor of 6, the choice would be mostly arbitrary. Though even in that case, choosing few would be more accurate, because converting  $4 \times \pi$  to few  $\times$  few underestimated the answer, meaning that 6 should be converted to 10 to compensate.

Such fine distinctions are, however, not needed in this problem because it has two factors of 6. Let's therefore convert one 6 to few and the other 6 to 10. In other words:

$$6 \times 6 \rightarrow \text{few} \times 10 \approx 30.$$

Putting it all together, including the 12 powers of 10, the surface area becomes

$$A \sim \underbrace{\text{few}}_4 \times \underbrace{\text{few}}_\pi \times \underbrace{\text{few}}_6 \times \underbrace{10}_6 \times 10^{12} \text{ m}^2 \sim 3 \cdot 10^{14} \text{ m}^2,$$

where the three factors of few turned into one factor of 10 and one few (roughly 3).

- c.  $10! \equiv 10 \times 9 \times \dots \times 1$ . The actual product is 3,628,800.

First convert each factor using factors of few and 10:

$$10 \rightarrow 10;$$

$$9 \rightarrow 10;$$

$$8 \rightarrow 10;$$

$$7 \rightarrow 10;$$

$$6 \rightarrow ??;$$

$$5 \rightarrow \text{few};$$

$$4 \rightarrow \text{few};$$

$$3 \rightarrow \text{few};$$

$$2 \rightarrow \text{few};$$

$$1 \rightarrow 1;$$

The conversion for 6 is blank for the moment because it could go either to few or 10: Unlike the product in **Problem 1.a**, the  $10!$  product contains only one factor of 6.

So think about whether the other conversions over- or underestimate the product. The conversions for 7, 8, and 9 are all overestimates. The conversions for 4 and 5 are underestimates but are perhaps balanced by the overestimate of converting 2 to few. So I reckon that, aside from the 6 conversion, the conversions overall overestimate the product. Therefore, I convert 6 to few rather than to 10.

Then  $10!$  becomes:

$$10 \times 10 \times 10 \times 10 \times \text{few} \times \text{few} \times \text{few} \times \text{few} \times \text{few} \times 1 = \text{few} \cdot 10^6 \approx 3 \cdot 10^6,$$

quite close to the exact value of 3,628,800.

## 2. Two-to-one-odds ranges

Give your two-to-one-odds range for the following quantities *without* doing a divide-and-conquer estimate:

- a. Mass of a full 747 jumbo jet. In **Problem 4** you use divide and conquer to make a more precise estimate.

The 2–1 odds range is the range that I’m reasonably confident of, that I’d feel some surprise if it were wrong. Before I do divide and conquer, this range is wide.

If I learnt that the true mass was below  $10^4$  kg, I’d be very surprised based on a rough feel for multi-ton masses. For example, an SUV might have a mass of a few tons ( $3 \cdot 10^3$  kg), and road signs warn about not driving trucks of more than a few tons over certain bridges. So my moderate-surprise value is higher than  $10^4$  kg, say  $3 \cdot 10^4$  kg.

On the high end, if I learnt that the mass was above  $10^6$  kg, I’d be slightly surprised. So my 2-to-1 range is  $3 \cdot 10^4 \dots 10^6$  kg.

- b. Mass of air in the 6.055J/2.038J classroom. In **Problem 5** you use divide and conquer to make a more precise estimate, and then you evaluate the precision of the new estimate.

The mass of a 747 was difficult to estimate because it’s so large, and we don’t have much practice with large masses. The mass of air in a room is more manageable, but estimating it is difficult for an alternative reason: We never lift air with our hands, so we don’t have a feel for its weight.

My range for the air mass is therefore wide. Let me think about the endpoints. Hmm, if I learnt that  $m$  was less than 1 kg, I’d be somewhat surprised. If I learnt that it was more than 1000 kg, I’d also be somewhat surprised. So my mass range is  $1 \dots 1000$  kg.

## 3. Combining plausible ranges

If the plausible range for  $x$  and for  $y$  is  $1 \dots 10$  (and  $x$  and  $y$  are independent), what is the midpoint of the plausible range for the product  $xy$ ? (The midpoint of a plausible range is the geometric mean of its endpoints.) What is the plausible range for  $xy$ ?

The midpoint of a range is the geometric mean of the endpoints, so the midpoint of each range is  $\sqrt{1 \times 10} = \sqrt{10}$ . The midpoint of the range for  $xy$  is the product of the midpoints, so it is  $\sqrt{10} \times \sqrt{10} = 10$ .

To find the plausible range for  $xy$ , first find the ratio of its endpoints. This ratio is computed from the endpoint ratio for  $x$  (which is  $r_x = 10$ ) and the ratio for  $y$  (which is  $r_y = 10$ ). Since the combinations happen on a logarithmic scale, what matters is their logarithms  $\log r_x$  and  $\log r_y$ .

In a terrible world, the logarithms would just add to give  $\log r_{xy}$ :

$$\log r_x + \log r_y = \log r_{xy}.$$

That procedure is equivalent to multiplying endpoints.

But that procedure is too pessimistic. The correct formula, based on the log-normal distribution, takes into account that errors are not likely to all point in the same direction. It says:

$$(\log r_x)^2 + (\log r_y)^2 = (\log r_{xy})^2.$$

What kind of logarithm (base 10, natural, base 2, etc.) does not matter as long as all terms use the same base (show this!). So I'll choose base 10. Then  $\log_{10} r_x = 1$  and  $\log_{10} r_y = 1$ , and each term on the left side is 1. So  $\log_{10} r_{xy} = \sqrt{2}$ , and

$$r_{xy} = 10^{\sqrt{2}} \approx 10^{1.4} \approx 50.$$

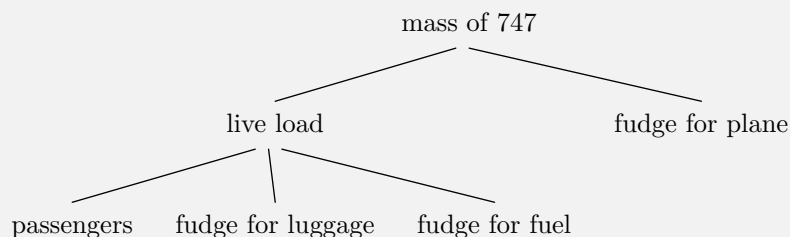
Therefore the ratio of the upper to lower endpoints for  $xy$  is 25. The midpoint is halfway between the two endpoints, so each endpoint is a factor of  $\sqrt{25} = 5$  away from the midpoint. Since the midpoint is 10, the range for  $xy$  is roughly 2...50.

## Problems

### 4. 747

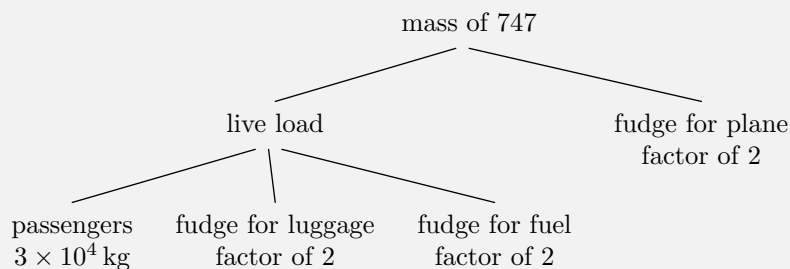
Estimate the mass of a full 747 jumbo jet, explaining your estimate using a tree. Then compare data online against your estimate from this problem and from [Problem 2.a](#). We'll use the mass later in the course when we estimate how much energy it costs to fly.

One method – and as usual there are many methods – is to estimate the mass of the passengers and then fudge that value to include the rest of the load (baggage and fuel), and then fudge that value to include the mass of the empty plane. Here is a tree to represent the method:



Now let's put values at the leaves. The plane holds about 400 passengers each weighing 70 kg, so  $3 \times 10^4$  kg. The fuel may be a factor of 2 or 3 larger than the passengers with luggage, although I am not confident about this number. And the plane itself, when empty, might weigh as much as all the load, so include another factor of 2 for the plane itself.

The tree with leaf values is



Now propagate values upward. The result is

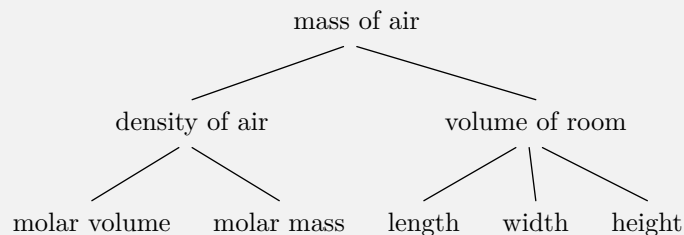
$$m \sim 3 \times 10^4 \text{ kg} \times 2 \times 2 \times 2 \sim 2.5 \times 10^5 \text{ kg}.$$

The actual maximum takeoff weight is (from Wikipedia)  $4 \times 10^5$  kg. Not bad, and more accurate than an initial guess without subdividing would have been!

## 5. Air mass

- a. Use divide-and-conquer to estimate the mass of air in the 6.055J/2.038J classroom and explain your estimate with a tree. If you have not yet seen the classroom, try harder to attend lecture!

One way to estimate the mass is to subdivide into the volume of the room and the density of air. The volume of the room subdivides into its length, height, and width. I remember that the density of air is roughly  $1 \text{ g } \ell^{-1}$  because the value is needed often in estimation problems. Alternatively, you can use a useful fact from chemistry, that one mole of an ideal gas at standard temperature and pressure occupies 22 liters, and combine that fact with the molar mass of air. Using that method, the tree is

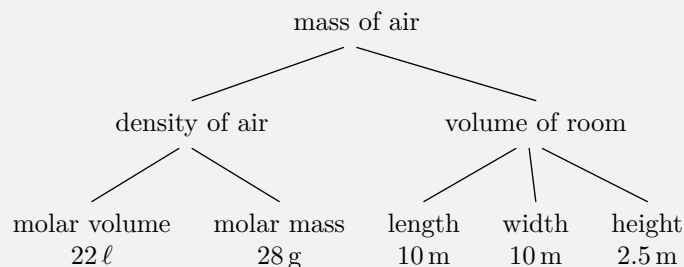


Now put values at the leaves.

For the room dimensions, the MIT schedules office webpage gives the room area, but let's estimate the dimensions by eye. Most rooms are 8 or 9 feet high, or about 2.5 m. The room has about 10 rows, spaced around 1 m apart. So the length is about 10 m. The room is roughly square in aspect ratio, so the width is around 10 m as well. [The MIT classrooms page says that the area is 825 square feet, or about  $77 \text{ m}^2$ . The estimate of  $100 \text{ m}^2$  is reasonably accurate.]

The molar volume for air (like any ideal gas) is 22 liters. The molar mass is, roughly, the molar mass of nitrogen, so about 14 g. But wait, nitrogen is diatomic, so the molar mass is 28 g.

The tree with values is:



Now propagate values upward. The volume of the room is  $250 \text{ m}^3$ . The density of air is roughly  $28/22 \text{ g } \ell^{-1}$ , or roughly  $1 \text{ kg } \text{m}^{-3}$ . So the mass of air in the room is roughly  $250 \text{ kg}$  or about 500 pounds.

- b. Give your plausible ranges (your 2-to-1-odds ranges) for the leaf nodes in your tree, and use those ranges to estimate your plausible range for the mass of air in the room.

Here are my leaf quantities and ranges:

molar volume	20...25 ℓ
molar mass	28...30 g
length	7 ... 14 m
width	7 ... 14 m
height	2 ... 3 m

Here are some internal reasonings that I used to produce those ranges. For the molar volume, I remember it as 22 ℓ, but I don't remember the temperature – e.g. room temperature or at 0°C? So I put in a fudge of around 10% above and below 22 ℓ to account for the uncertainty. For the molar mass, I remember that air is mostly nitrogen, which is diatomic (N<sub>2</sub>), and I remember that nitrogen has an atomic mass of 14, so 28 g is a reasonable molar mass for air. But air has lots of other gases; the rest is mostly oxygen, with a higher atomic mass. Rather than working out the weighted average of oxygen and nitrogen's atomic masses, I lazily put in an uncertainty factor around the molar mass using 28...30 g.

The ratio (of upper to lower endpoints) at the root node results from the ratios at the leaf nodes. Those ratios are given in the last column:

molar volume	20...25 ℓ	1.25
molar mass	28...30 g	1.07
length	7 ... 14 m	2
width	7 ... 14 m	2
height	2 ... 3 m	1.5

I could calculate the resulting ratio using the `combine.py` script, but it's easier to eyeball it. The two ratios of 2 make a ratio of  $2^{\sqrt{2}}$  – for the same reason that, in [Problem 3](#), two ratios of 10 produce a ratio of  $10^{\sqrt{2}}$ . And  $2^{\sqrt{2}}$  is roughly 2.5. The ratio of 1.25 and 1.5 combine to make, let's say, 1.6. The ratio of 1.07, being so nearly 1, hardly contributes anything to the final ratio.

Now combine the ratio of 2.5 and 1.6. If the ratios were 2.5 and 2.5, the result would be  $2.5^{\sqrt{2}}$ , or about 3.5. But 1.6 is somewhat less than 2.5, so maybe the final ratio is 3.0 instead. (An exact calculation using the formulas or the program gives 2.96...)

So 250 kg – the divide-and-conquer estimate – is the midpoint of a range with ratio roughly 3. The endpoints are therefore  $250 \text{ kg}/\sqrt{3}$  and  $250 \text{ kg} \times \sqrt{3}$ , or roughly 150...450 kg.

- c. Compare this new plausible range with the initial range you gave in [Problem 2.b](#).

My original range for the mass of air ([Problem 2.b](#)) was 1...1000 kg, which has a ratio of 1000. Ack! But after doing divide and conquer, my range shrunk to 150...450 kg. Most of the reduction from a ratio of a factor of 1000 to a ratio of factor of 3 comes from replacing a hard-to-guess quantity – the air mass – with easier-to-guess quantities, such as the room dimensions.

In addition, some reduction comes from the expectation of error cancellation. Naively multiplying the ratios of the leaf ranges gives the overly pessimistic ratio of roughly 8, whereas the log-normal rule of combination gave a ratio of 3. So, error-cancellation helped us by a factor of 8/3 (roughly 3), and dividing into easy-to-estimate quantities lowered the ratio by a factor of 125 (from 1000 to 8).

**6. Your turn to create**

Invent – but do not solve! – an estimation question that divide and conquer would help solve. To give you an idea for the kind of problems that work well, the classic of this genre, due to Fermi, is ‘How many piano tuners are there in Chicago?’ Other examples are the problems that we solved in lecture.

Particularly interesting or instructive questions might appear on the website or as examples in lecture or the notes (let me know should you *not* want your name attributed in case your question gets selected).