6.055J/2.038J (Spring 2009)

Homework 4

Do the following warmups and problems. Due in class on ***Monday, 4 May 2009****.*

Open universe: *Collaboration, notes, and other sources of information are* ***encouraged.*** *However, avoid looking up answers until you solve the problem (or have tried hard). That policy helps you learn the most from the problems.*

Homework will be graded with a light touch: P (made a reasonable effort), D (did not make a reasonable effort), or F (did not turn in).

If you have a thesis due on Friday 8 May, you can take an automatic extension until Monday 11 May (there will likely be a final problem set due on 13 May, handed out at least a week before that date).

Warmups

1. **Counting dimensionless groups**
   How many independent dimensionless groups are there in the following sets of variables:

   a. size of hydrogen including relativistic effects:

   $e^2/4\pi\epsilon_0$, $\hbar$, c, a_0 (Bohr radius), m_e (electron mass).

   b. period of a spring–mass system in a gravitational field:

   T (period), k (spring constant), m, x_0 (amplitude), g.

   c. speed at which a free-falling object hits the ground:

   v, g, h (initial drop height).

   d. [tricky!] weight W of an object:

   W, g, m.

2. **Integrals by dimensions**
   You can use dimensions to do integrals. As an example, try this integral:

$$I(\beta) = \int_{-\infty}^{\infty} e^{-\beta x^2}\,dx.$$

   Which choice has correct dimensions:

(a.) $\sqrt{\pi}\beta^{-1}$ **(b.)** $\sqrt{\pi}\beta^{-1/2}$ **(c.)** $\sqrt{\pi}\beta^{1/2}$ **(d.)** $\sqrt{\pi}\beta^{1}$

Hints:

1. The dimensions of dx are the same as the dimensions of x.

2. Pick interesting dimensions for x, such as length. (If x is dimensionless then you cannot use dimensional analysis on the integral.)

Problems

3. How to avoid remembering lots of constants

Many atomic problems, such as the size or binding energy of hydrogen, end up in expressions with $\hbar$, the electron mass m_e, and $e^2/4\pi\epsilon_0$, which is a nicer way to express the squared electron charge. You can avoid having to remember those constants if instead you remember these values instead:

$$\begin{aligned} \hbar c &\approx 200\,\text{eV nm} = 2000\,\text{eV Å} \\ m_e c^2 &\sim 0.5 \cdot 10^6\,\text{eV} \\ \frac{e^2/4\pi\epsilon_0}{\hbar c} &\equiv \alpha \approx \frac{1}{137} \quad \text{(fine-structure constant).} \end{aligned}$$

Use those values to evaluate the Bohr radius in angstroms (1 Å = 0.1 nm):

$$a_0 = \frac{\hbar^2}{m_e(e^2/4\pi\epsilon_0)}.$$

As an example calculation using the $\hbar c$ value, here is the energy of a photon:

$$E = hf = 2\pi\hbar f = 2\pi\hbar\frac{c}{\lambda},$$

where f is its frequency and λ is its wavelength. For green light, $\lambda \sim 600$ nm, so

$$E \sim \frac{\overbrace{6}^{2\pi} \times \overbrace{200\,\text{eV nm}}^{\hbar c}}{\underbrace{600\,\text{nm}}_{\lambda}} \sim 2\,\text{eV}.$$

4. Heavy nuclei

In lecture we analyzed hydrogen, which is one electron bound to one proton. In this problem you study the innermost electron in an atom such as uranium that has many protons, and analyze one physical consequence of its binding energy.

So, imagine a nucleus with Z protons around which orbits one electron. Let $E(Z)$ be the binding energy (the hydrogen energy is the case $Z = 1$).

a. Show that the ratio $E(Z)/E(1)$ is Z^2.

b. In lecture, we derived that $E(1)$ is the kinetic energy of an electron moving with speed αc where α is the fine-structure constant (roughly 10^{-2}). How fast does the innermost electron move around a heavy nucleus with charge Z?

c. When that speed is comparable to the speed of light, the electron has a kinetic energy comparable to its (relativistic) rest energy. One consequence of such a high kinetic energy is that the electron has enough kinetic energy to produce a positron (an anti-electron) out of nowhere ('pair creation'). That positron leaves the nucleus, turning a proton into a neutron as it exits. So the atomic number Z drops by one: The nucleus is unstable! Relativity sets an upper limit for Z.

Estimate that maximum Z and compare it with the Z for the heaviest stable nucleus (uranium).

5. Power radiated by an accelerating charge

Electromagnetism, where the usual derivations are so cumbersome, is an excellent area to apply dimensional analysis. In this problem you work out the power radiated by an accelerating charge, which is how radio stations work.

So, consider a particle with charge q, with position x, velocity v, and acceleration a. What variables are relevant to the radiated power P? The position cannot matter because it depends on the origin of the coordinate system, whereas the power radiated cannot depend on the origin. The velocity cannot matter because of relativity: You can transform to a reference frame where $v = 0$, but that change will not affect the radiation (otherwise you could distinguish a moving frame from a non-moving frame, in violation of the principle of relativity). So the acceleration a is all that's left to determine the radiated power. [This line of argument is slightly dodgy, but it works for low speeds.]

a. Using P, $q^2/4\pi\epsilon_0$, and a, how many dimensionless quantities can you form?

b. Fix the problem in the previous part by adding one quantity to the list of variables, and give a physical reason for including the quantity.

c. With the new list, use dimensionless groups to find the power radiated by an accelerating point charge. In case you are curious, the exact result contains a dimensionless factor of $2/3$; dimensional analysis triumphs again!

6. Yield from an atomic bomb

Geoffrey Taylor, a famous Cambridge fluid mechanic, annoyed the US government by doing the following analysis. The question he answered: 'What was the yield (in kilotons of TNT) of the first atomic blast (in the New Mexico desert in 1945)?' Declassified pictures, which even had a scale bar, gave the following data on the radius of the explosion at various times:

t (ms)	R (m)
3.26	59.0
4.61	67.3
15.0	106.5
62.0	185.0

a. Use dimensional analysis to work out the relation between radius R, time t, blast energy E, and air density ρ.

b. Use the data in the table to estimate the blast energy E (in Joules).

c. Convert that energy to kilotons of TNT. One gram of TNT releases 1 kcal or roughly 4 kJ.

The actual value was 20 kilotons, a classified number when Taylor published his result ['The Formation of a Blast Wave by a Very Intense Explosion. II. The Atomic Explosion of 1945.', *Proceedings of the Royal Society of London. Series A, Mathematical and Physical* **201**(1065): 175–186 (22 March 1950)]

7. Your turn to create
Invent – but you do not need to solve! – an estimation question that dimensional analysis would help solve.

Optional

8. Atomic blast: A physical interpretation
Use energy densities and sound speeds to make a rough physical explanation of the result in the 'yield from an atomic bomb' problem.