the contact force from the track. In choosing the shape of the track, you affect the contact force on the roller coaster, and thereby its acceleration, velocity, and position. There are an infinity of possible tracks, and we do not want to analyze each one to find the forces and acceleration. An invariant, energy, simplifies the analysis. No matter what tricks the track does, the kinetic plus potential energy

$$\frac{1}{2}mv^2 + mgh$$

is constant. The roller coaster starts with $v = 0$ and height h_{start}; it can never rise above that height without violating the constancy of the energy. The invariant – the conserved quantity – solves the problem in one step, avoiding an endless analysis of an infinity of possible paths.

The moral of this section is: *When there is change, look for what does not change.*

6.2 Flight

How far can birds and planes fly? The theory of flight is difficult and involves vortices, Bernoulli's principle, streamlines, and much else. This section offers an alternative approach: use conservation estimate the energy required to generate lift, then minimize the lift and drag contributions to the energy to find the minimum-energy way to make a trip.

6.2.1 Lift

Instead of wading into the swamp of vortices, study what does not change. In this case, the vertical component of the plane's momentum does not change while it cruises at constant altitude.

Because of momentum conservation, a plane must deflect air downward. If it did not, gravity would pull the plane into the ground. By deflecting air downwards – which generates lift – the plane gets a compensating, upward recoil. Finding the necessary recoil leads to finding the energy required to produce it.

Imagine a journey of distance s. I calculate the energy to produce lift in three steps:

1. How much air is deflected downward?
2. How fast must that mass be deflected downward in order to give the plane the needed recoil?
3. How much kinetic energy is imparted to that air?

The plane is moving forward at speed v, and it deflects air over an area L^2 where L is the wingspan. Why this area L^2, rather than the cross-sectional area, is subtle. The reason is that the wings disturb the flow over a distance comparable to their span (the longest length). So when the plane travels a distance s, it deflects a mass of air

$$m_{\text{air}} \sim \rho L^2 s.$$

The downward speed imparted to that mass must take away enough momentum to compensate for the downward momentum imparted by gravity. Traveling a distance s takes time s/v, in which time gravity imparts a downward momentum Mgs/v to the plane. Therefore

$$m_{\text{air}} v_{\text{down}} \sim \frac{Mgs}{v}$$

so

$$v_{\text{down}} \sim \frac{Mgs}{v m_{\text{air}}} \sim \frac{Mgs}{\rho v L^2 s} = \frac{Mg}{\rho v L^2}.$$

The distance s divides out, which is a good sign: The downward velocity of the air should not depend on an arbitrarily chosen distance!

The kinetic energy required to send that much air downwards is $m_{\text{air}} v_{\text{down}}^2$. That energy factors into $(m_{\text{air}} v_{\text{down}}) v_{\text{down}}$, so

$$E_{\text{lift}} \sim \underbrace{m_{\text{air}} v_{\text{down}}}_{Mgs/v} v_{\text{down}} \sim \frac{Mgs}{v} \underbrace{\frac{Mg}{\rho v L^2}}_{v_{\text{down}}} = \frac{(Mg)^2}{\rho v^2 L^2} s.$$

Check the dimensions: The numerator is a squared force since Mg is a force, and the denominator is a force, so the expression is a force times the distance s. So the result is an energy.

Interestingly, the energy to produce lift decreases with increasing speed. Here is a scaling argument to make that result plausible. Imagine doubling the speed of the plane. The fast plane makes the journey in one-half the time of the original plane. Gravity has only one-half the time to pull the plane down, so the plane needs only one-half the recoil to stay aloft. Since the same mass of air is being deflected downward but with half the total recoil (momentum), the necessary downward velocity is a factor of 2 lower for the fast plane than for the slow plane. This factor of 2 in speed lowers the energy by a factor of 4, in accordance with the v^{-2} in E_{lift}.

6.2.2 Optimization including drag

The energy required to fly includes the energy to generate lift and to fight drag. I'll add the lift and drag energies, and choose the speed that minimizes the sum.

The energy to fight drag is the drag force times the distance. The drag force is usually written as

$$F_{\text{drag}} \sim \rho v^2 A,$$

where A is the cross-sectional area. The missing dimensionless constant is $c_{\text{d}}/2$:

$$F_{\text{drag}} = \frac{1}{2} c_{\text{d}} \rho v^2 A,$$

where c_d is the drag coefficient.

However, to simplify comparing the energies required for lift and drag, I instead write the drag force as

$$F_{drag} = C\rho v^2 L^2,$$

where C is a modified drag coefficient, where the drag is measured relative to the squared wingspan rather than to the cross-sectional area. For most flying objects, the squared wingspan is much larger than the cross-sectional area, so C is much smaller than c_d.

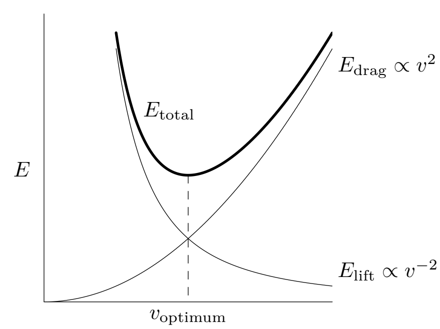
With that form for F_{drag}, the drag energy is

$$E_{drag} = C\rho v^2 L^2 s,$$

and the total energy to fly is

$$E \sim \underbrace{\frac{(Mg)^2}{\rho v^2 L^2} s}_{E_{lift}} + \underbrace{C\rho v^2 L^2 s}_{E_{drag}}.$$

A sketch of the total energy versus velocity shows interesting features. At low speeds, lift is the dominant consumer because of its v^{-2} dependence. At high speeds, drag is the dominant consumer because of its v^2 dependence. In between these extremes is an optimum speed $v_{optimum}$: the speed that minimizes the energy consumption for a fixed journey distance s. Going faster or slower than the optimum speed means consuming more energy. That extra consumption cannot always be avoided. A plane is designed so that its cruising speed is its minimum-energy speed. So at takeoff and landing, when its speed is much less than the minimum-energy speed, a plane requires a lot of power to stay aloft, which is one reason that the engines are so loud at takeoff and landing (another reason is probably that the engine noise reflects off the ground and back to the plane).



The constraint, or assumption, that a plane travels at the minimum-energy speed simplifies the expression for the total energy. At the minimum-energy speed, the drag and lift energies are equal. So

$$\frac{(Mg)^2}{\rho v^2 L^2} s \sim C\rho v^2 L^2 s,$$

or

$$Mg \sim C^{1/2} \rho v^2 L^2.$$

This constraint simplifies the total energy. Instead of simplifying the sum, simplify just the drag, which neglects only a factor of 2 since drag and lift are roughly equal at the minimum-energy speed. So

$$E \sim E_{\text{drag}} \sim C\rho v^2 L^2 s \sim C^{1/2} Mgs.$$

This result depends in reasonable ways upon M, g, C, and s. First, lift overcomes gravity, and gravity produces the plane's weight Mg. So Mg should show up in the energy, and the energy should, and does, increase when Mg increases. Second, a streamlined plane should use less energy than a bluff, blocky plane, so the energy should, and does, increase as the modified drag coefficient C increases. Third, since the flight is at a constant speed, the energy should be, and is, proportional to the distance traveled s.

6.2.3 How the maximum range depends on size

Before calculating a range for a particular plane or bird, evaluate the scaling: How does the range depend on the size of the plane? As for the mountain-height analysis (**Section 5.2**), assume that all planes are geometrically similar (have the same shape) and therefore differ only in size.

Since the energy required to fly a distance s is $E \sim C^{1/2}Mgs$, a tank of fuel gives a range of

$$s \sim \frac{E_{\text{tank}}}{C^{1/2}Mg}.$$

Let β be the fuel fraction: the fraction of the plane's mass taken up by fuel. Then $M\beta$ is the fuel mass, and $M\beta\mathcal{E}$ is the energy contained in the fuel, where $\mathcal{E}$ is the energy density (energy per mass) of the fuel. With that notation, $E_{\text{tank}} \sim M\beta\mathcal{E}$ and

$$s \sim \frac{M\beta\mathcal{E}}{C^{1/2}Mg} = \frac{\beta\mathcal{E}}{C^{1/2}g}.$$

Since all planes, at least in this analysis, have the same shape, their modified drag coefficient C is also the same. And all planes face the same gravitational field strength g. So the denominator is the same for all planes. The numerator contains β and $\mathcal{E}$. Both parameters are the same for all planes. So the numerator is the same for all planes. Therefore

$$s \propto 1.$$

All planes can fly the same distance!

Even more surprising is to apply this reasoning to migrating birds. Here is the ratio of ranges:

$$\frac{s_{\text{plane}}}{s_{\text{bird}}} \sim \frac{\beta_{\text{plane}}}{\beta_{\text{bird}}} \frac{\mathcal{E}_{\text{plane}}}{\mathcal{E}_{\text{bird}}} \left(\frac{C_{\text{plane}}}{C_{\text{bird}}}\right)^{-1/2}.$$

Take the factors in turn. First, the fuel fraction β_{plane} is perhaps 0.3 or 0.4. The fuel fraction β_{bird} is probably similar: A well-fed bird having fed all summer is perhaps 30 or 40% fat. So $\beta_{\text{plane}}/\beta_{\text{bird}} \sim 1$. Second, jet fuel energy density is similar to fat's energy density, and plane engines and animal metabolism are comparably efficient (about 25%). So $\mathcal{E}_{\text{plane}}/\mathcal{E}_{\text{bird}} \sim 1$. Finally, a bird has a similar shape to a plane – it is not a great approximation, but it has the virtue of simplicity. So $C_{\text{bird}}/C_{\text{plane}} \sim 1$.

Therefore, planes and well-fed, migrating birds should have the same maximum range! Let's check. The longest known nonstop flight by an animal is 11,570 km, made by a bar-tailed godwit from Alaska to New Zealand (tracked by satellite). The maximum range for a 747-400 is 13,450 km, only slightly longer than the godwit's range.

6.2.4 Explicit computations

To get an explicit range, not only how the range scales with size, estimate the fuel fraction β, the energy density $\mathcal{E}$, and the drag coefficient C. For the fuel fraction I'll guess $\beta \sim 0.4$. For $\mathcal{E}$, look at the nutrition label on the back of a pack of butter. Butter is almost all fat, and one serving of 11 g provides 100 Cal (those are 'big calories'). So its energy density is $9\,\mathrm{kcal\,g^{-1}}$. In metric units, it is $4 \cdot 10^7\,\mathrm{J\,kg^{-1}}$. Including a typical engine efficiency of one-fourth gives

$$\mathcal{E} \sim 10^7\,\mathrm{J\,kg^{-1}}.$$

The modified drag coefficient needs converting from easily available data. According to Boeing, a 747 has a drag coefficient of $C' \approx 0.022$, where this coefficient is measured using the wing area:

$$F_{\text{drag}} = \frac{1}{2} C' A_{\text{wing}} \rho v^2.$$

Alas, this formula is a third convention for drag coefficients, depending on whether the drag is referenced to the cross-sectional area A, wing area A_{wing}, or squared wingspan L^2.

It is easy to convert between the definitions. Just equate the standard definition

$$F_{\text{drag}} = \frac{1}{2} C' A_{\text{wing}} \rho v^2.$$

to our definition

$$F_{\text{drag}} = C L^2 \rho v^2$$

to get

$$C = \frac{1}{2} \frac{A_{\text{wing}}}{L^2} C' = \frac{1}{2} \frac{l}{L} C',$$

since $A_{\text{wing}} = Ll$ where l is the wing width. For a 747, $l \sim 10\,\mathrm{m}$ and $L \sim 60\,\mathrm{m}$, so $C \sim 1/600$.

Combine the values to find the range:

$$s \sim \frac{\beta \mathcal{E}}{C^{1/2} g} \sim \frac{0.4 \times 10^7\,\mathrm{J\,kg^{-1}}}{(1/600)^{1/2} \times 10\,\mathrm{m\,s^{-2}}} \sim 10^7\,\mathrm{m} = 10^4\,\mathrm{km}.$$

The maximum range of a 747-400 is 13,450 km. The maximum known nonstop flight by a bird – indeed, by any animal – is 11,570 km: A female bar-tailed godwit tracked by satellite migrated between Alaska and New Zealand. The approximate analysis of the range is unreasonably accurate.

Next I estimate the minimum-energy speed and compare it to the cruising speed of a 747. The sum of drag and lift energies is a minimum when the speed is given by