Since A is the cross-sectional area of the animal, Ah is the volume of air that it sweeps out in the jump, and ρAh is the mass of air swept out in the jump. So the relative importance of drag has a physical interpretation as a ratio of the mass of air displaced to the mass of the animal.

To find how this ratio depends on animal size, rewrite it in terms of the animal's side length l. In terms of side length, $A \sim l^2$ and $m \propto l^3$. What about the jump height h? The simplest analysis predicts that all animals have the same jump height, so $h \propto l^0$. Therefore the numerator ρAh is $\propto l^1$, the denominator m is $\propto l^3$, and

$$\frac{E_{\text{drag}}}{E_{\text{required}}} \propto \frac{l^2}{l^3} = l^{-1}.$$

So, small animals have a large ratio, meaning that drag affects the jumps of small animals more than it affects the jumps of large animals. The missing constant of proportionality means that we cannot say at what size an animal becomes 'small' for the purposes of drag. So the calculation so far cannot tell us whether fleas are included among the small animals.

The jump data, however, substitutes for the missing constant of proportionality. The ratio is

$$\frac{E_{\text{drag}}}{E_{\text{required}}} \sim \frac{\rho Ah}{m} \sim \frac{\rho l^2 h}{\rho_{\text{animal}} l^3}.$$

It simplifies to

$$\frac{E_{\text{drag}}}{E_{\text{required}}} \sim \frac{\rho}{\rho_{\text{animal}}} \frac{h}{l}.$$

As a quick check, verify that the dimensions match. The left side is a ratio of energies, so it is dimensionless. The right side is the product of two dimensionless ratios, so it is also dimensionless. The dimensions match.

Now put in numbers. A density of air is $\rho \sim 1\,\text{kg}\,\text{m}^{-3}$. The density of an animal is roughly the density of water, so $\rho_{\text{animal}} \sim 10^3\,\text{kg}\,\text{m}^{-3}$. The typical jump height – which is where the data substitutes for the constant of proportionality – is $60\,\text{cm}$ or roughly $1\,\text{m}$. A flea's length is about $1\,\text{mm}$ or $l \sim 10^{-3}\,\text{m}$. So

$$\frac{E_{\text{drag}}}{E_{\text{required}}} \sim \frac{1\,\text{kg}\,\text{m}^{-3}}{10^3\,\text{kg}\,\text{m}^{-3}} \frac{1\,\text{m}}{10^{-3}\,\text{m}} \sim 1.$$

The ratio being unity means that if a flea would jump to $60\,\text{cm}$, overcoming drag would require roughly as much as energy as would the jump itself in vacuum.

Drag provides a plausible explanation for why fleas do not jump as high as the typical height to which larger animals jump.

5.4.3 Cycling

This section discusses cycling as an example of how drag affects the performance of people as well as fleas. Those results will be used in the analysis of swimming, the example of the next section.

What is the world-record cycling speed? Before looking it up, predict it using armchair proportional reasoning. The first task is to define the kind of world record. Let's say that the cycling is on a level ground using a regular bicycle, although faster speeds are possible using special bicycles or going downhill.

To estimate the speed, make a model of where the energy goes. It goes into rolling resistance, into friction in the chain and gears, and into drag. At low speeds, the rolling resistance and chain friction are probably important. But the importance of drag rises rapidly with speed, so at high-enough speeds, drag is the dominant consumer of energy.

For simplicity, assume that drag is the only consumer of energy. The maximum speed happens when the power supplied by the rider equals the power consumed by drag. The problem therefore divides into two estimates: the power consumed by drag and the power that an athlete can supply.

The drag power P_{drag} is related to the drag force:

$$P_{\text{drag}} = F_{\text{drag}}v \sim \rho v^3 A.$$

It indeed rises rapidly with velocity, supporting the initial assumption that drag is the important effect at world-record speeds.

Setting $P_{\text{drag}} = P_{\text{athlete}}$ gives

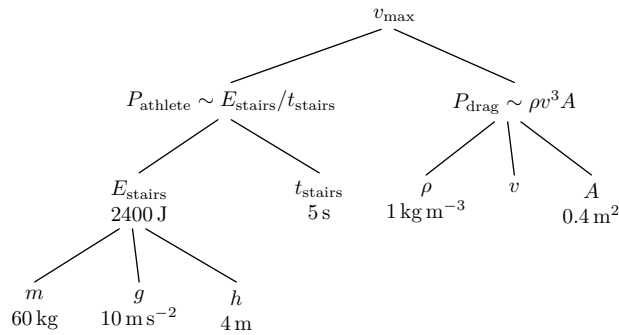$$v_{\text{max}} \sim \left(\frac{P_{\text{athlete}}}{\rho A}\right)^{1/3}$$

To estimate how much power an athlete can supply, I ran up one flight of stairs leading from the MIT Infinite Corridor. The Infinite Corridor, being an old building, has spacious high ceilings, so the vertical climb is perhaps $h \sim 4\,\text{m}$ (a typical house is $3\,\text{m}$ per storey). Leaping up the stairs as fast as I could, I needed $t \sim 5\,\text{s}$ for the climb. My mass is $60\,\text{kg}$, so my power output was

$$
\begin{aligned}
P_{\text{author}} &\sim \frac{\text{potential energy supplied}}{\text{time to deliver it}} \\
&= \frac{mgh}{t} \sim \frac{60\,\text{kg} \times 10\,\text{m}\,\text{s}^{-2} \times 4\,\text{m}}{5\,\text{s}} \sim 500\,\text{W}.
\end{aligned}
$$

P_{athlete} should be higher than this peak power since most authors are not Olympic athletes. Fortunately I'd like to predict the endurance record. An Olympic athlete's long-term power might well be comparable to my peak power. So I use $P_{\text{athlete}} = 500\,\text{W}$.

The remaining item is the cyclist's cross-sectional area A. Divide the area into width and height. The width is a body width, perhaps $0.4\,\text{m}$. A racing cyclist crouches, so the height is maybe $1\,\text{m}$ rather than a full $2\,\text{m}$. So $A \sim 0.4\,\text{m}^2$.

Here is the tree that represents this analysis:

Now combine the estimates to find the maximum speed. Putting in numbers gives

$$v_{max} \sim \left(\frac{P_{athlete}}{\rho A}\right)^{1/3} \sim \left(\frac{500\,\text{W}}{1\,\text{kg m}^{-3} \times 0.4\,\text{m}^2}\right)^{1/3}.$$

The cube root might suggest using a calculator. However, massaging the numbers simplifies the arithmetic enough to do it mentally. If only the power were 400 W or, instead, if the area were 0.5 m! Therefore, in the words of Captain Jean-Luc Picard, 'make it so'. The cube root becomes easy:

$$v_{max} \sim\sim \left(\frac{400\,\text{W}}{1\,\text{kg m}^{-3} \times 0.4\,\text{m}^2}\right)^{1/3} \sim (1000\,\text{m}^3\,\text{s}^{-3})^{1/3} = 10\,\text{m s}^{-1}.$$

So the world record should be, if this analysis has any correct physics in it, around $10\,\text{m s}^{-1}$ or 22 mph.

The world one-hour record – where the contestant cycles as far as possible in one hour – is 49.7 km or 30.9 mi. The estimate based on drag is reasonable!

5.4.4 Swimming

The last section's analysis of cycling helps predict the world-record speed for swimming. The last section showed that

$$v_{max} \sim \left(\frac{P_{athlete}}{\rho A}\right)^{1/3}.$$

To evaluate the maximum speed for swimming, one could put in a new ρ and A directly into that formula. However, that method replicates the work of multiplying, dividing, and cube-rooting the various values.

Instead it is instructive to scale the numerical result for cycling by looking at how the maximum speed depends on the parameters of the situation. In other words, I'll use the formula for v_{max} to work out the ratio $v_{swimmer}/v_{cyclist}$, and then use that ratio along with $v_{cyclist}$ to work out $v_{swimmer}$.

The speed v_{max} is

$$v_{max} \sim \left(\frac{P_{athlete}}{\rho A}\right)^{1/3}.$$

So the ratio of swimming and cycling speeds is

$$\frac{v_{\text{swimmer}}}{v_{\text{cyclist}}} \sim \left(\frac{P_{\text{swimmer}}}{P_{\text{cyclist}}}\right)^{1/3} \times \left(\frac{\rho_{\text{swimmer}}}{\rho_{\text{cyclist}}}\right)^{-1/3} \times \left(\frac{A_{\text{swimmer}}}{A_{\text{cyclist}}}\right)^{-1/3}.$$

Estimate each factor in turn. The first factor accounts for the relative athletic prowess of swimmers and cyclists. Let's assume that they generate equal amounts of power; then the first factor is unity. The second factor accounts for the differing density of the mediums in which each athlete moves. Roughly, water is 1000 times denser than air. So the second factor contributes a factor of 0.1 to the speed ratio. If the only factors were the first two, then the swimming world record would be about $1 \, \text{m s}^{-1}$.

Let's compare with reality. The actual world record for a 1500-m freestyle (in a 50-m pool) is 14m34.56s set in July 2001 by Grant Hackett. That speed is $1.713 \, \text{m s}^{-1}$, significantly higher than the prediction of $1 \, \text{m s}^{-1}$.

The third factor comes to the rescue by accounting for the relative profile of a cyclist and a swimmer. A swimmer and a cyclist probably have the same width, but the swimmer's height (depth in the water) is perhaps one-sixth that of a crouched cyclist. So the third factor contributes $6^{1/3}$ to the predicted speed, making it $1.8 \, \text{m s}^{-1}$.

This prediction is close to the actual record, closer to reality than one might expect given the approximations in the physics, the values, and the arithmetic. However, the accuracy is a result of the form of the estimate, that the maximum speed is proportional to the cube root of the athlete's power and the inverse cube root of the cross-sectional area. Errors in either the power or area get compressed by the cube root. For example, the estimate of $500 \, \text{W}$ might easily be in error by a factor of 2 in either direction. The resulting error in the maximum speed is $2^{1/3}$ or 1.25, an error of only 25%. The cross-sectional area of a swimmer might be in error by a factor of 2 as well, and this mistake would contribute only a 25% error to the maximum speed. [With luck, the two errors would cancel!]

5.4.5 Flying

In the next example, I scale the drag formula to estimate the fuel efficiency of a jumbo jet. Rather than estimating the actual fuel consumption, which would produce a large, meaningless number, it is more instructive to estimate the relative fuel efficiency of a plane and a car.

Assume that jet fuel goes mostly to fighting drag. This assumption is not quite right, so at the end I'll discuss it and other troubles in the analysis. The next step is to assume that the drag force for a plane is given by the same formula as for a car:

$$F_{\text{drag}} \sim \rho v^2 A.$$

Then the ratio of energy consumed in travelling a distance d is

$$\frac{E_{\text{plane}}}{E_{\text{car}}} \sim \frac{\rho_{\text{up-high}}}{\rho_{\text{low}}} \times \left(\frac{v_{\text{plane}}}{v_{\text{car}}}\right)^2 \times \frac{A_{\text{plane}}}{A_{\text{car}}} \times \frac{d}{d}.$$

Estimate each factor in turn. The first factor accounts for the lower air density at a plane's cruising altitude. At 10 km, the density is roughly one-third of the sea-level density, so the first factor contributes 1/3. The second factor accounts for the faster speed of a plane. Perhaps $v_{\text{plane}} \sim 600$ mph and $v_{\text{car}} \sim 60$ mph, so the second factor contributes a factor of 100. The third factor accounts for the greater cross-sectional area of the plane. As a reasonable estimate

$$A_{\text{plane}} \sim 6\,\text{m} \times 6\,\text{m} = 36\,\text{m}^2,$$

whereas

$$A_{\text{car}} \sim 2\,\text{m} \times 1.5\,\text{m} = 3\,\text{m}^2,$$

so the third factor contributes a factor of 12. The fourth factor contributes unity, since we are analyzing the plane and car making the same trip (New York to Los Angeles, say).

The result of the four factors is

$$\frac{E_{\text{plane}}}{E_{\text{car}}} \sim \frac{1}{3} \times 100 \times 12 \sim 400.$$

A plane looks incredibly inefficient. But I neglected the number of people. A jumbo jet takes carries 400 people; a typical car, at least in California, carries one person. So the plane and car come out equal!

This analysis leaves out many effects. First, jet fuel is used to generate lift as well as to fight drag. However, as a later analysis will show, the energy consumed in generating lift is comparable to the energy consumed in fighting drag. Second, a plane is more streamlined than a car. Therefore the missing constant in the drag force $F_{\text{drag}} \sim \rho v^2 A$ is smaller for a plane than for a car. our crude analysis of drag has not included this effect. Fortunately this error compensates, or perhaps overcompensates, for the error in neglecting lift.

5.5 Analysis of algorithms

Proportional reasoning is the basis of an entire subject of the analysis of algorithms, a core part of computer science. How fast does an algorithm run? How much space does it require? A proportional-reasoning analysis helps you decide which algorithms to use. This section discusses these decisions using the problem of how to square very large numbers.

Squaring numbers is a special case of multiplication, but the algebra is simpler for squaring than for multiplying since having only one number as the input means there are fewer variables in the analysis.

Here is a divide-and-conquer version of the standard school multiplication algorithm.