Chapter 4
Symmetry

Symmetry is often thought of as a purely geometric concept, but it is useful in a wide variety of problems. Whenver you can use symmetry, use it and will simplify the solution. The following sections illustrate symmetry in calculus, geometry, and heat transfer.

4.1 Calculus

For what value of x is $3x - x^2$ a maximum?

The usual method is to take the derivative:

$$\frac{d}{dx}(3x - x^2) = 3 - 2x = 0,$$

whereupon $x_{\text{max}} = 3/2$.

Although differentiating is a general method, its generality comes at a cost: that its results are often hard to interpret. One does the manipulations, and whatever formulas show up at the end, so be it. So, if you can find a simplification, you are likely to get a more insight into why the answer came out the way that it did.

For this problem, symmetry simplifies it enough that nothing remains to do. To see how, first factor the equation into $x(3 - x)$. Let x_{max} be where it has its maximum. The factors x and $3 - x$ can be swapped using the substitution $x' = 3 - x$. In terms of x', the problem becomes maximizing $(3 - x')x'$. This formula has the same structure as the original one $x(3 - x)$! So the symmetry operation preserves this structure. Since the x or x' location of the maximum depends only on the structure, the location has the same numerical value whether in the x or x' coordinate systems. So it is said to be invariant under the substitution operation. Therefore, in this problem, the $x' \to 3 - x$ substitution is a symmetry.

Since $x' = 3 - x$ and, as a result of symmetry, $x'_{\text{max}} = x_{\text{max}}$, the only solution is $x_{\text{max}} = x'_{\text{max}} = 3/2$.

A similar, perhaps more telegraphic argument, is that the maximum is halfway between the two roots $x = 0$ and $x = 3$, so the maximum is, again, at $x_{\text{max}} = 3/2$. This argument implicitly contains symmetry, which is the justification for saying that the maximum is midway between the roots.