

Chapter 5

Proportional reasoning

Symmetry wrings out excess, irrelevant complexity, and proportional reasoning in one implementation of that philosophy. If an object moves with no forces on it (or if you walk steadily), then moving for twice as long means doubling the distance traveled. Having two changing quantities contributes complexity. However, the ratio distance/time, also known as the speed, is independent of the time. It is therefore simpler than distance or time. This conclusion is perhaps the simplest example of proportional reasoning, where the proportional statement is

$$\text{distance} \propto \text{time}.$$

Using symmetry has mitigated complexity. Here the symmetry operation is ‘change for how long the object move (or how long you walk)’. This operation should not change conclusions of an analysis. So, do the analysis using quantities that themselves are unchanged by this symmetry operation. One such quantity is the speed, which is why speed is such a useful quantity.

Similarly, in random walks and diffusion problems, the mean-square distance traveled is proportional to the time travelled:

$$\langle x^2 \rangle \propto t.$$

So the interesting quantity is one that does not change when t changes:

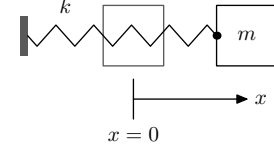
$$\text{interesting quantity} \equiv \frac{\langle x^2 \rangle}{t}.$$

This quantity is so important that it is given a name – the diffusion constant – and is tabulated in handbooks of material properties.

5.1 Period of a spring–mass system

As a first example of proportional reasoning, here is one way to explain a famous result in physics: that the period of spring–mass system is independent of the amplitude.

So imagine a mass m connected to the wall by a spring with spring constant k . If disturbed, the mass oscillates. The period of the system is the time for the mass to make a round trip through the equilibrium position.



Extend the spring by a distance x_0 ; this displacement is the amplitude. To see how it affects the period, make an approximation, which will be an example of throwing away information (the topic of **Part 3**). The approximation is to pretend that the pendulum moves with a constant speed v . Then the period is

$$T \sim \frac{\text{distance}}{\text{speed } v},$$

and the distance that the mass travels in one period is $4x_0$. Ignore the factor of 4:

$$T \sim \frac{x_0}{v}.$$

Proportional reasoning helps us estimate v by an energy argument. The initial potential energy is $\text{PE} \sim kx_0^2$ or

$$\text{PE} \propto x_0^2.$$

The maximum kinetic energy, which we use as a proxy for the typical kinetic energy, is the initial potential energy, so

$$\text{KE}_{\text{typical}} \propto x_0^2$$

as well. The typical velocity is $\sqrt{\text{KE}_{\text{typical}}}$, so

$$v_{\text{typical}} \propto x_0.$$

That result is great news because it means that the period is proportional to 1:

$$T \propto \frac{x_0}{x_0} = x_0^0.$$

In other words, the period is independent of amplitude.

5.2 Mountain heights

The next example of proportional reasoning explains why mountains cannot become too high. Assume that all mountains are cubical and made of the same material. Making that assumption discards actual complexity, the topic of **Part 3**. However, it is a useful approximation.

To see what happens if a mountain gets too large, estimate the pressure at the base of the mountain. Pressure is force divided by area, so estimate the force and the area.

The area is the easier estimate. With the approximation that all mountains are cubical and made of the same kind of rock, the only parameter distinguishing one mountain from another is its side length l . The area of the base is then l^2 .