Chapter 2

Assorted subproblems

For the first example of dividing into unlike subproblems, we estimate the spacing between pits on a CD ROM. Then we estimate the amount of oil that the United States imports annually.

2.1 Pits on a CDROM

Q: What is the spacing between the pits on a CDROM? The pits (indentations) are the memory elements, each pit storing one bit of information.

A quick estimate comes from turning over a CDROM and enjoying the brilliant colors. The colors arise because the arrangement of pits diffracts visible light by a significant angle, and the angle depends strongly on the wavelength (or color). So the pits are spaced comparably to the wavelength of light, say about 1 μm.

A second estimate might come from knowing a bit about the laser in a CD player or in a CDROM drive. It is a near-infrared laser, so its wavelength – which will be comparable to the pit size and spacing – is slightly longer than visible-light wavelengths. Since visible-light wavelengths range from 350 to 700 nm or from 0.4 to 0.8 μm, a reasonable estimate for the pit spacing is again 1 μm.

These two estimates agree, which increases our confidence in each estimate. Here is why. Because the methods are so different, an error in one method is likely to be significantly different from an error in the second method. Therefore, if the estimates agree, they are probably both reasonable. The lesson is to use as many diverse methods as you can.

The third method uses divide-and-conquer reasoning. The capacity and area together determine the pit spacing, if we make the useful approximation that the pits are regularly spaced. [This approximation is an example of discarding information, which is the extended topic of **Part 3**.]

The area is $A \sim (10 \, \text{cm})^2$.

The capacity is often on the box: 640 MB, which is about 5 gigabits since each byte is 8 bits. After including error-correcting bits, perhaps the capacity is 6 or 7 gigabits.

The pit spacing d comes from arranging those $N \sim 10$ gigabits into a regular lattice of bits:

$$d \sim \sqrt{\frac{A}{N}} \sim \frac{10\,\text{cm}}{10^5} \sim 1\,\mu\text{m}.$$

Once again, the estimate is around $1\,\mu\text{m}$. Any result that we derive three times has to be true!

You do not need to take the capacity figure on faith. Instead, use divide-and-conquer reasoning based on how much information would be needed to encode the music on an audio CD.

The information needed depends on the play time, the sampling rate, and the sample size (number of bits).

A typical CD holds about 20 popular-music songs, each about 3 minutes long, so it is about 1 hour. An alternative piece of (perhaps bogus) history confirms this estimate: The engineers at Philips who invented the CD format and player allegedly insisted that the format hold Beethoven's Ninth Symphony, around 74 minutes.

The sampling rate is $44\,\text{kHz}$. Suppose you had remembered the 44 but did not remember the units: whether they were kHz or MHz. How do you choose? Human hearing extends to about $20\,\text{kHz}$. For comparison, the $60\,\text{Hz}$ line-voltage hum is quite well into the audible range. Lossless sampling of sound, according to Shannon's sampling theorem, needs to have a rate of at least $2 \times 20\,\text{kHz}$. The inventors of the CD format chose a slightly higher rate, so that one can make a half-decent anti-alias filter. [If you want to know more about anti-alias filters, let me know!] Even the constraint of an anti-alias filter does not require a sampling rate of $44\,\text{MHz}$. Indeed, the sampling rate is $44\,\text{kHz}$.

Each sample requires 32 bits: two channels (stereo) each needing 16 bits per sample. The 16 bits is a reasonable compromise between the utopia of exact volume encoding (∞ bits per sample per channel) and the utopia of minimal storage (1 bit per sample per channel). Why compromise at 16 bits rather than, say, 50 bits? Because 50 bits, while easy nowadays to represent digitally, implies absurd analog hardware that has an accuracy of 1 part in 2^{50}.

So the capacity is roughly

$$N \sim 1\,\text{hours} \times \frac{3600\,\text{s}}{1\,\text{hr}} \times \frac{4.4 \times 10^4\,\text{samples}}{1\,\text{s}} \times \frac{2 \times 16\,\text{bits}}{1\,\text{sample}}.$$

First do the important part: the powers of ten. The 3600 contributes three; the 4.4×10^4 contributes four; and the 2×16 contributes one; for a total of eight.

The mantissas – the parts in front of the power of ten – contribute $3.6 \times 4.4 \times 3.3$. This multiplication is simplified if you remember that there are only two numbers in the world: 1 and 'few'. The only rule to remember is that (few$^2 = 10$, so 'few' acts a lot like 3. Then 3.6×3.3 is roughly 10, perhaps a bit higher. Then $3.6 \times 4.4 \times 3.3 \sim 50$.

So the estimate for the capacity is roughly $50 \times 10^8 \sim 5 \cdot 10^9$, which agrees amazingly well with the figure from a box of CDROM's. Therefore, the divide-and-conquer estimate for the capacity gives us even more confidence in our estimate for the pit spacing.