The volume of the pyramid is $V \sim hb^2$, and the missing constant must make volume 4/3. Since $hb^2 = 4$ for these pyramids, the missing constant is 1/3. Voilà:

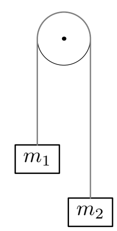
$$V = \frac{1}{3}hb^2 = \frac{4}{3}.$$

8.2 Mechanics

8.2.1 Atwood machine

The next problem illustrates dimensional analysis and special cases in a physical problem. Many of the ideas and methods from the geometry example transfer to this problem, and it introduces more methods and ways of reasoning.

The problem is a staple of first-year physics: Two masses, m_1 and m_2, are connected and, thanks to a pulley, are free to move up and down. What is the acceleration of the masses and the tension in the string? You can solve this problem with standard methods from first-year physics, which means that you can can check the solution that we derive using dimensional analysis, educated guessing, and a feel for functions.



The first problem is to find the acceleration of, say, m_1. Since m_1 and m_2 are connected by a rope, the acceleration of m_2 is, depending on your sign convention, either equal to m_1 or equal to $-m_1$. Let's call the acceleration a and use dimensional analysis to guess its form. The first step is to decide what variables are relevant. The acceleration depends on gravity, so g should be on the list. The masses affect the acceleration, so m_1 and m_2 are on the list. And that's it. You might wonder what happened to the tension: Doesn't it affect the acceleration? It does, but it is itself a consequence of m_1, m_2, and g. So adding tension to the list does not add information; it would instead make the dimensional analysis difficult.

Var	*Dim*	What
a	LT^{-2}	accel. of m_1
g	LT^{-2}	gravity
m_1	M	block mass
m_2	M	block mass

These variables fall into two pairs where the variables in each pair have the same dimensions. So there are two dimensionless groups here ripe for picking: $G_1 = m_1/m_2$ and $G_2 = a/g$. You can make any dimensionless group using these two obvious groups, as experimentation will convince you. Then, following the usual pattern,

$$\frac{a}{g} = f\left(\frac{m_1}{m_2}\right),$$

where f is a dimensionless function.

Pause a moment. The more thinking that you do to choose a clean representation, the less algebra you do later. So rather than find f using m_1/m_2 as the dimensionless group, first choose a better group. The ratio m_1/m_2 does not respect the symmetry of the problem in that only the sign of the acceleration changes when you interchange the labels m_1 and m_2. Whereas m_1/m_2 turns into its reciprocal. So the function f will have to do lots of work to turn the unsymmetric ratio m_1/m_2 into a symmetric acceleration.

Back to the drawing board for how to fix G_1. Another option is to use $m_1 - m_2$. Wait, the difference is not dimensionless! I fix that problem in a moment. For now observe the virtue of $m_1 - m_2$. It shows a physically reasonable symmetry under mass interchange: $G_1 \to -G_1$. To make it dimensionless, divide it by another mass. One candidate is m_1:

$$G_1 = \frac{m_1 + m_2}{m_1}.$$

That choice, like dividing by m_2, abandons the beloved symmetry. But dividing by $m_1 + m_2$ solves all the problems:

$$G_1 = \frac{m_1 - m_2}{m_1 + m_2}.$$

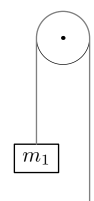
This group is dimensionless and it respects the symmetry of the problem.

Using this G_1, the solution becomes

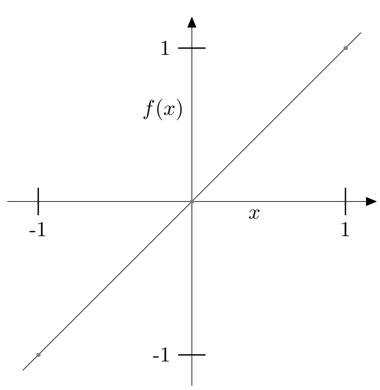
$$\frac{a}{g} = f\left(\frac{m_1 - m_2}{m_1 + m_2}\right),$$

where f is another dimensionless function.

To guess $f(x)$, where $x = G_1$, try special cases. First imagine that m_1 becomes huge. A quantity with mass cannot be huge on its own, however. Here huge means *huge relative to* m_2, whereupon $x \approx 1$. In this thought experiment, m_1 falls as if there were no m_2 so $a = -g$. Here we've chosen a sign convention with positive acceleration being upward. If m_2 is huge relative to m_1, which means $x = -1$, then m_2 falls like a stone pulling m_1 upward with acceleration $a = g$. A third limiting case is $m_1 = m_2$ or $x = 0$, whereupon the masses are in equilibrium so $a = 0$.



Here is a plot of our knowledge of f:



The simplest conjecture – an educated guess – is that $f(x) = x$. Then we have our result:

$$\frac{a}{g} = \frac{m_1 - m_2}{m_1 + m_2}.$$

Look how simple the result is when derived in a symmetric, dimensionless form using special cases!