The other possibility is to add quantum mechanics, which was developed to solve funda-
mental problems like the existence of matter. The physics of quantum mechanics is com-
plicated, but its effect on dimensional analyses is simple: It contributes a new constant of
nature $\hbar$ whose dimensions are those of angular momentum. Angular momentum is mvr,
so

$$[\hbar] = \mathrm{ML}^2\mathrm{T}^{-1}.$$

The $\hbar$ might save the day. There are now two quantities
containing time dimensions. Since $e^2/4\pi\epsilon_0$ has T^{-2} and
$\hbar$ has T^{-1}, the ratio $\hbar^2/(e^2/4\pi\epsilon_0)$ contains no time dimen-
sions. Since

Var	Dim	What
a_0	L	size
$e^2/4\pi\epsilon_0$	$\mathrm{ML}^3\mathrm{T}^{-2}$	
m_e	M	electron mass
$\hbar$	$\mathrm{ML}^2\mathrm{T}^{-1}$	quantum

$$\left[\frac{\hbar^2}{e^2/4\pi\epsilon_0}\right] = \mathrm{ML},$$

a dimensionless group is

$$\frac{\hbar^2}{a_0 m_e(e^2/4\pi\epsilon_0)}$$

It turns out that all dimensionless groups can be formed from this group. So, as in the
spring–mass example, the only possible true statement involving this group is

$$\frac{\hbar^2}{a_0 m_e(e^2/4\pi\epsilon_0)} = \text{dimensionless constant.}$$

Therefore, the size of hydrogen is

$$a_0 \sim \frac{\hbar^2}{m_e(e^2/4\pi\epsilon_0)}.$$

Putting in values for the constants gives

$$a_0 \sim 0.5\text{Å} = 0.5\cdot 10^{-10}\,\mathrm{m}.$$

It turns out that the missing dimensionless constant is 1, so the dimensional analysis has
given the exact answer.

7.4.2 Atomic sizes and substance densities

Hydrogen has a diameter of 1Å. A useful consequence is the rule of thumb is that a typical
interatomic spacing is 3Å. This approximation gives a reasonable approximation for the
densities of substances, as this section explains.

Let A be the atomic mass of the atom; it is (roughly) the number of pro-
tons and neutrons in the nucleus. Although A is called a mass, it is
dimensionless. Each atom occupies a cube of side length $a \sim 3\,\text{Å}$, and
has mass $A m_{\text{proton}}$. The density of the substance is

$$\rho = \frac{\text{mass}}{\text{volume}} \sim \frac{A m_{\text{proton}}}{(3\,\text{Å})^3}.$$

You do not need to remember or look up m_{proton} if you multiply this
fraction by unity in the form of N_A/N_A, where N_A is Avogadro's number:

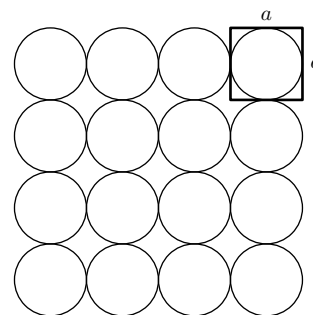$$\rho \sim \frac{A m_{\text{proton}} N_A}{(3\,\text{Å})^3 \times N_A}.$$

The numerator is $A\,\text{g}$, because that is how N_A is defined. The denominator is

$$3 \cdot 10^{-23}\,\text{cm}^3 \times 6 \cdot 10^{23} = 18.$$

So instead of remembering m_{proton}, you need to remember N_A. However, N_A is more famil-
iar than m_{proton} because N_A arises in chemistry and physics. Using N_A also emphasizes the
connection between microscopic and macroscopic values. Carrying out the calculations:

$$\rho \sim \frac{A}{18}\,\text{g cm}^{-3}.$$

The table compares the estimate against reality. Most every-
day elements have atomic masses between 15 and 150, so the
density estimate explains why most densities lie between 1 and
$10\,\text{g cm}^{-3}$. It also shows why, for materials physics, cgs units
are more convenient than SI units are. A typical cgs density of
a solid is $3\,\text{g cm}^{-3}$, and 3 is a modest number and easy to re-
member and work with. However, a typical SI density of a solid
$3000\,\text{kg m}^{-3}$. Numbers such as 3000 are unwieldy. Each time
you use it, you have to think, 'How many powers of ten were
there again?' So the table tabulates densities using the cgs units

Element	$\rho_{\text{estimated}}$	ρ_{actual}
Li	0.39	0.54
H_2O	1.0	1.0
Si	1.56	2.4
Fe	3.11	7.9
Hg	11.2	13.5
Au	10.9	19.3
U	13.3	18.7

of g cm^{-3}. I even threw a joker into the pack – water is not an element! – but the density
estimate is amazingly accurate.

7.4.3 Physical interpretation

The previous method, dimensional analysis, is mostly mathematical. As a second computa-
tion of a_0, we show you a method that is mostly physics. Besides checking the Bohr radius,
it provides a physical interpretation of it. The Bohr radius is the radius of the orbit with the
lowest energy (the ground state). The energy is a sum of kinetic and potential energy. This
division suggests, again, a divide-and-conquer approach: first the kinetic energy, then the
potential energy.