

Whereas using dimensions reveals that the problem is simpler: to find the function  $h$  such that

$$\frac{kT^2}{m} = h().$$

Here  $h()$  means a function of no variables. Why no variables? Because the right side contains all the other quantities on which  $kT^2/m$  could depend. However, dimensional analysis says that the variables appear only through the combination  $kT^2/m$ , which is already on the left side. So no variables remain to be put on the right side; hence  $h$  is a function of zero variables. The only function of zero variables is a constant, so  $kT^2/m = C$ .

This pattern illustrates a famous quote from the statistician and physicist Harold Jeffreys [6, p. 82]:

A good table of functions of one variable may require a page; that of a function of two variables a volume; that of a function of three variables a bookcase; and that of a function of four variables a library.

Use dimensions; avoid tables as big as a library!

## 7.4 Hydrogen atom

Hydrogen is the simplest atom, and studying hydrogen is the simplest way to understand the **atomic theory**. Feynman has explained the importance of the atomic theory in his famous lectures on physics [7, p. 1-2]:

If, in some cataclysm, all of scientific knowledge were to be destroyed, and only one sentence passed on to the next generations of creatures, what statement would contain the most information in the fewest words? I believe it is the *atomic hypothesis* (or the *atomic fact*, or whatever you wish to call it) that *all things are made of atoms – little particles that move around in perpetual motion, attracting each other when they are a little distance apart, but repelling upon being squeezed into one another*. In that one sentence, you will see, there is an *enormous* amount of information about the world. . .

The atomic theory was first stated by Democritus. (Early Greek science and philosophy is discussed with wit, sympathy, and insight in Bertrand Russell's *History of Western Philosophy* [8].) Democritus could not say much about the properties of atoms. With modern knowledge of classical and quantum mechanics, and dimensional analysis, you can say more.

### 7.4.1 Dimensional analysis

The next example of dimensional reasoning is the hydrogen atom in order to answer two questions. The first question is how big is it. That size sets the size of more complex atoms and molecules. The second question is how much energy is needed to disassemble hydrogen. That energy sets the scale for the bond energies of more complex substances, and

those energies determine macroscopic quantities like the stiffness of materials, the speed of sound, and the energy content of fat and sugar. All from hydrogen!

The first step in a dimensional analysis is to choose the relevant variables. A simple model of hydrogen is an electron orbiting a proton. The orbital force is provided by electrostatic attraction between the proton and electron. The magnitude of the force is

$$\frac{e^2}{4\pi\epsilon_0} \frac{1}{r^2}$$

where  $r$  is the distance between the proton and electron. The list of variables should include enough variables to generate this expression for the force. It could include  $q$ ,  $\epsilon_0$ , and  $r$  separately. But that approach is needlessly complex: The charge  $q$  is relevant only because it produces a force. So the charge appears only in the combined quantity  $e^2/4\pi\epsilon_0$ . A similar argument applies to  $\epsilon_0$ .

Therefore rather than listing  $q$  and  $\epsilon_0$  separately, list only  $e^2/4\pi\epsilon_0$ . And rather than listing  $r$ , list  $a_0$ , the common notation for the Bohr radius (the radius of ideal hydrogen). The acceleration of the electron depends on the electrostatic force, which can be constructed from  $e^2/4\pi\epsilon_0$  and  $a_0$ , and on its mass  $m_e$ . So the list should also include  $m_e$ . To find the dimensions of  $e^2/4\pi\epsilon_0$ , use the formula for force

$$F = \frac{e^2}{4\pi\epsilon_0} \frac{1}{r^2}$$

Then

$$\left[ \frac{e^2}{4\pi\epsilon_0} \right] = [r^2] \times [F] = \text{ML}^3\text{T}^{-2}$$

The next step is to make dimensionless groups. However, no combination of these three items is dimensionless. To see why, look at the time dimension because it appears in only one quantity,  $e^2/4\pi\epsilon_0$ . So that quantity cannot occur in a dimensionless group: If it did, there would be no way to get rid of the time dimensions. From the two remaining quantities,  $a_0$  and  $m_e$ , no dimensionless group is possible.

The failure to make a dimensionless group means that hydrogen does not exist in the simple model as we have formulated it. I neglected important physics. There are two possibilities for what physics to add.

One possibility is to add relativity, encapsulated in the speed of light  $c$ . So we would add  $c$  to the list of variables. That choice produces a dimensionless group, and therefore produces a size. However, the size is not the size of hydrogen. It turns out to be the classical electron radius instead. Fortunately, you do not have to know what the classical electron radius is in order to understand why the resulting size is not the size of hydrogen. Adding relativity to the physics – or adding  $c$  to the list – allows radiation. So the orbiting, accelerating electron would radiate. As radiation carries energy away from the electron, it spirals into the proton, meaning that in this world hydrogen does not exist, nor do other atoms.

| <i>Var</i>       | <i>Dim</i>            | What $\omega$ |
|------------------|-----------------------|---------------|
| T <sup>-1</sup>  | <i>frequency</i>      | k             |
| L <sup>-1</sup>  | <i>wavenumber</i>     | g             |
| LT <sup>-2</sup> | <i>gravity</i>        | h             |
| L                | <i>depth</i>          | $\rho$        |
| ML <sup>-3</sup> | <i>density</i>        | $\gamma$      |
| MT <sup>-2</sup> | <i>surfacetension</i> |               |

The other possibility is to add quantum mechanics, which was developed to solve fundamental problems like the existence of matter. The physics of quantum mechanics is complicated, but its effect on dimensional analyses is simple: It contributes a new constant of nature  $\hbar$  whose dimensions are those of angular momentum. Angular momentum is  $mvr$ , so

$$[\hbar] = \text{ML}^2\text{T}^{-1}.$$

The  $\hbar$  might save the day. There are now two quantities containing time dimensions. Since  $e^2/4\pi\epsilon_0$  has  $T^{-2}$  and  $\hbar$  has  $T^{-1}$ , the ratio  $\hbar^2/(e^2/4\pi\epsilon_0)$  contains no time dimensions. Since

$$\left[ \frac{\hbar^2}{e^2/4\pi\epsilon_0} \right] = \text{ML},$$

a dimensionless group is

$$\frac{\hbar^2}{a_0 m_e (e^2/4\pi\epsilon_0)}$$

It turns out that all dimensionless groups can be formed from this group. So, as in the spring-mass example, the only possible true statement involving this group is

$$\frac{\hbar^2}{a_0 m_e (e^2/4\pi\epsilon_0)} = \text{dimensionless constant.}$$

Therefore, the size of hydrogen is

$$a_0 \sim \frac{\hbar^2}{m_e (e^2/4\pi\epsilon_0)}.$$

Putting in values for the constants gives

$$a_0 \sim 0.5\text{\AA} = 0.5 \cdot 10^{-10} \text{ m.}$$

It turns out that the missing dimensionless constant is 1, so the dimensional analysis has given the exact answer.

#### 7.4.2 Atomic sizes and substance densities

Hydrogen has a diameter of  $1\text{\AA}$ . A useful consequence is the rule of thumb is that a typical interatomic spacing is  $3\text{\AA}$ . This approximation gives a reasonable approximation for the densities of substances, as this section explains.

| <i>Var</i>           | <i>Dim</i>                 | What          |
|----------------------|----------------------------|---------------|
| $a_0$                | L                          | size          |
| $e^2/4\pi\epsilon_0$ | $\text{ML}^3\text{T}^{-2}$ |               |
| $m_e$                | M                          | electron mass |
| $\hbar$              | $\text{ML}^2\text{T}^{-1}$ | quantum       |