

7.3 Dimensionless groups

Dimensionless ratios are useful. For example, in the oil example, the ratio of the two quantities has dimensions; in that case, the dimensions of the ratio are time (or one over time). If the authors of the article had used a dimensionless ratio, they might have made a valid comparison.

This section explains why dimensionless ratios are the only quantities that you need to think about; in other words, that there is no need to think about quantities with dimensions.

To see why, take a concrete example: computing the energy E to produce lift as a function of distance traveled s , plane speed v , air density ρ , wingspan L , plane mass m , and strength of gravity g . Any true statement about these variables looks like

$$\triangle_{\text{mess}} + \square_{\text{mess}} = \circ_{\text{mess}},$$

where the various messes mean 'a horrible combination of E , s , v , ρ , L , and m .

As horrible as that true statement is, it permits the following rewriting: Divide each term by the first one (the triangle). Then

$$\frac{\triangle_{\text{mess}}}{\triangle_{\text{mess}}} + \frac{\square_{\text{mess}}}{\triangle_{\text{mess}}} = \frac{\circ_{\text{mess}}}{\triangle_{\text{mess}}},$$

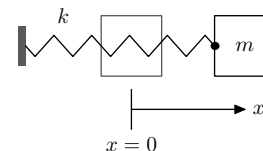
The first ratio is 1, which has no dimensions. Without knowing the individual messes, we don't know the second ratio; but it has no dimensions because it is being added to the first ratio. Similarly, the third ratio, which is on the right side, also has no dimensions.

So the rewritten expression is dimensionless. Nothing in the rewriting depended on the particular form of the true statement, except that each term has the same dimensions.

Therefore, *any true statement can be rewritten in dimensionless form.*

Dimensionless forms are made from dimensionless ratios, so all you need are dimensionless ratios, and you can do all your thinking with them. Here is a familiar example to show how this change simplifies your thinking. This example uses familiar physics so that you can concentrate on the new idea of dimensionless ratios.

The problem is to find the period of an oscillating spring–mass system given an initial displacement x_0 , then allowed to oscillate freely. [Section 5.1](#) gave a proportional-reasoning analysis of this system. The relevant variables that determine the period T are mass m , spring constant k , and amplitude x_0 . Those three variables completely describe the system, so any true statement about period needs only those variables.



Since any true statement can be written in dimensionless form, the next step is to find all dimensionless forms that can be constructed from T , m , k , and x_0 . A table of dimensions is helpful. The only tricky entry is the dimensions of a spring constant. Since the force from the spring is $F = kx$, where x is the displacement, the dimensions of a spring constant are the dimensions of force divided by the dimensions of x . It is convenient to have a notation for the concept of ‘the dimensions of’. In that notation,

<i>Var</i>	<i>Dim</i>	What
T	T	period
m	M	mass
k	MT^{-2}	spring constant
x_0	L	amplitude

$$[k] = \frac{[F]}{[x]},$$

where [quantity] means the dimensions of the quantity. Since $[F] = MLT^{-2}$ and $[x] = L$,

$$[k] = MT^{-2},$$

which is the entry in the table.

These quantities combine into many – infinitely many – dimensionless combinations or groups:

$$\frac{kT^2}{m}, \frac{m}{kT^2}, \left(\frac{kT^2}{m}\right)^{25}, \pi \frac{m}{kT^2}, \dots$$

The groups are redundant. You can construct them from only one group. In fancy terms, all the dimensionless groups are formed from one *independent* dimensionless group. What combination to use for that one group is up to you, but you need only one group. I like kT^2/m .

So any true statement about the period can be written just using kT^2/m . That requirement limits the possible statements to

$$\frac{kT^2}{m} = C,$$

where C is a dimensionless constant. This form has two important consequences:

1. The amplitude x_0 does not affect the period. This independence is also known as simple harmonic motion. The analysis in [Section 5.1](#) gave an approximate argument for why the period should be independent of the amplitude. So that approximate argument turns out to be an exact argument.
2. The constant C is independent of k and m . So I can measure it for one spring–mass system and know it for all spring–mass systems, no matter the mass or spring constant. The constant is a universal constant.

The requirement that dimensions be valid has simplified the analysis of the spring–mass system. Without using dimensions, the problem would be to find (or measure) the three-variable function f that connects m , k , and x_0 to the period:

$$T = f(m, k, x_0).$$

Whereas using dimensions reveals that the problem is simpler: to find the function h such that

$$\frac{kT^2}{m} = h().$$

Here $h()$ means a function of no variables. Why no variables? Because the right side contains all the other quantities on which kT^2/m could depend. However, dimensional analysis says that the variables appear only through the combination kT^2/m , which is already on the left side. So no variables remain to be put on the right side; hence h is a function of zero variables. The only function of zero variables is a constant, so $kT^2/m = C$.

This pattern illustrates a famous quote from the statistician and physicist Harold Jeffreys [6, p. 82]:

A good table of functions of one variable may require a page; that of a function of two variables a volume; that of a function of three variables a bookcase; and that of a function of four variables a library.

Use dimensions; avoid tables as big as a library!

7.4 Hydrogen atom

Hydrogen is the simplest atom, and studying hydrogen is the simplest way to understand the **atomic theory**. Feynman has explained the importance of the atomic theory in his famous lectures on physics [7, p. 1-2]:

If, in some cataclysm, all of scientific knowledge were to be destroyed, and only one sentence passed on to the next generations of creatures, what statement would contain the most information in the fewest words? I believe it is the *atomic hypothesis* (or the *atomic fact*, or whatever you wish to call it) that *all things are made of atoms – little particles that move around in perpetual motion, attracting each other when they are a little distance apart, but repelling upon being squeezed into one another*. In that one sentence, you will see, there is an *enormous* amount of information about the world. . .

The atomic theory was first stated by Democritus. (Early Greek science and philosophy is discussed with wit, sympathy, and insight in Bertrand Russell's *History of Western Philosophy* [8].) Democritus could not say much about the properties of atoms. With modern knowledge of classical and quantum mechanics, and dimensional analysis, you can say more.

7.4.1 Dimensional analysis

The next example of dimensional reasoning is the hydrogen atom in order to answer two questions. The first question is how big is it. That size sets the size of more complex atoms and molecules. The second question is how much energy is needed to disassemble hydrogen. That energy sets the scale for the bond energies of more complex substances, and