6.055J/2.038J (Spring 2008)

Solution set 6

*Do the following warmups and problems. Due in class on **Friday, 09 May 2008**.*

***Open universe:** Collaboration, notes, and other sources of information are **encouraged.** However, avoid looking up answers until you solve the problem (or have tried hard). That policy helps you learn the most from the problems.*

***Bring a photocopy to class on the due date**, trade it for a solution set, and figure out or ask me about any confusing points. Your work will be graded lightly: P (made a reasonable effort), D (did not make a reasonable effort), or F (did not turn in).*

Warmups

1. **Integrals**

   Use special cases of a to choose the correct formula for each integral.

   a. $\int_{-\infty}^{\infty} e^{-ax^2}\, dx$

   (1.) $\sqrt{\pi a}$ (2.) $\sqrt{\pi/a}$

   > The most useful special cases here are $a \to 0$ and $a \to \infty$. When a is zero, the Gaussian becomes the flat line $y = 1$, which has infinite area. The first choice, $\sqrt{\pi a}$, goes to zero in this limit, so it cannot be right. The second choice, $\sqrt{\pi/a}$, has the correct behavior.
   >
   > The limit $a \to \infty$ gives the same conclusion: The first choice cannot be right, and the second one might be right.

   b. $\int_{-\infty}^{\infty} \frac{1}{a^2 + x^2}\, dx$

   (1.) πa (2.) π/a (3.) $\sqrt{\pi a}$ (4.) $\sqrt{\pi/a}$

   > The easiest special case is $a \to \infty$. In that limit, the integrand is zero everywhere, so the integral is zero. The first and third choices are therefore incorrect.
   >
   > To decide between the second and fourth choices, use the special case $a = 1$. The integral becomes
   >
   > $$\int_{-\infty}^{\infty} \frac{1}{1 + x^2}\, dx$$
   >
   > The integral is $\arctan x$. At ∞ it contributes $\pi/2$, and at $-\infty$ it subtracts $-\pi/2$, so the integral is π. Only the second choice, π/a, has the correct behavior.
   >
   > [The problem statement had an error, which one of you found (thank you!): The problem should either have said $a \geq 0$ or have used $|a|$ in the candidate answers.]

2. **Debugging**

   Use special (i.e. easy) cases of n to decide which of these two C functions correctly computes the sum of the first n odd numbers:

   ```
   int sum_of_odds (int n) {
     int i, total = 0;
     for (i=1; i<=2*n+1; i+=2)
       total += i;
     return total;
   }
   ```

   or

   ```
   int sum_of_odds (int n) {
     int i, total = 0;
     for (i=1; i<=2*n-1; i+=2)
       total += i;
     return total;
   }
   ```

   > Special cases are useful in debugging programs. The easiest cases are often $n = 0$ or $n = 1$. Let's try $n = 0$ first. In the first program, the $2n + 1$ in the loop condition means that $i = 1$ is the only case, so the total becomes 1. Whereas the sum of the first 0 odd numbers should be zero! So the first program looks suspicious.
   >
   > Let's confirm that analysis using $n = 1$. The first program will have $i = 1$ and $i = 3$ in the loop, making the total $1 + 3 = 4$. The second program will have $i = 1$ in the loop, making the total 1. Since the correct answer is 1, the first program has a bug, and the second program looks sound.

3. **Reynolds numbers**

   Estimate the Reynolds number for:

   a. a falling raindrop;

   > From an earlier problem set, a raindrop falls at about $10\,\mathrm{m\,s^{-1}}$ and it has a radius of roughly $3\,\mathrm{mm}$. So the Reynolds number is
   >
   > $$\mathrm{Re} = \frac{rv}{\nu} \sim \frac{3 \cdot 10^{-3}\,\mathrm{m} \times 10\,\mathrm{m\,s^{-1}}}{1.5 \cdot 10^{-5}\,\mathrm{m^2\,s^{-1}}} \sim 2000.$$

   b. a flying mosquito;

   > Using reasonable guesses for the flight speed and size:
   >
   > $$\mathrm{Re} \sim \frac{10^{-3}\,\mathrm{m} \times 1\,\mathrm{m\,s^{-1}}}{10^{-5}\,\mathrm{m^2\,s^{-1}}} \sim 100.$$

> Even a mosquito experiences high-Reynolds-number drag. It's not quite turbulent flow, which happens around Re $\sim 10^3$, but it's still significantly higher than the value for low-Reynolds-number (Stokes) drag.

4. **Drag at low Reynolds number**

    At low Reynolds number, the drag on a sphere is

    $$F = 6\pi\rho\nu v r.$$

    What is the drag coefficient c_d as a function of Reynolds number Re?

    > The drag coefficient is
    >
    > $$c_\mathrm{d} \equiv \frac{F}{\frac{1}{2}\rho v^2 A},$$
    >
    > where $A = \pi r^2$ is the cross-sectional area. So
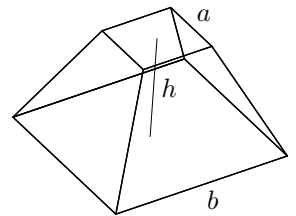   >
    > $$c_\mathrm{d} = \frac{6\pi\rho\nu v r}{\frac{1}{2}\rho v^2 \pi r^2} = 12\frac{\nu}{vr} = \frac{12}{Re}.$$

Problems

5. **Truncated pyramid**

    In this problem you use special cases to find the volume of a truncated pyramid. It has a square base with side b, a square top with side a, and height h. So, use special cases of a and b to evaluate these candidates for the volume:

    a. $\dfrac{1}{3}hb^2$

    b. $\dfrac{1}{3}ha^2$

    c. $\dfrac{1}{3}h(a^2 + b^2)$

    d. $\dfrac{1}{2}h(a^2 + b^2)$

    Which if any of these formulas pass all your special-cases tests? If no formula passes all tests, invent a formula that does. If you are stuck, find the volume by integration!

    > Three useful special cases are $a = 0$, $b = 0$, and $a = b$. Each special case is useful because it simplifies the figure. When $a = 0$, the figure is the original square-based pyramid with base side b. When $b = 0$, the figure is an inverted square-based pyramid with base side a. When $a = b$, the figure is a rectangular prism.
    >
    > The first candidate for the volume, $hb^2/3$, works when $a = 0$ but fails when $b = 0$ or $a = b$. The second candidate fails when $a = 0$ or $a = b$.

The third candidate works when $a = 0$ or $b = 0$. Great! However, it fails when $a = b$ because it predicts that the rectangular prism has volume $2hb^2/3$ rather than hb^2. In contrast, the fourth candidate works when $a = b$ but fails when $a = 0$ or $b = 0$.

So, we need a new formula. Making formulas just with a^2 and b^2 does not provide enough freedom to accommodate the three special cases. To think of another term, look at what a^2 and b^2 have in common. They are both quadratic. A third quadratic term, not yet used, is ab. So what about a formula like

$$V = \frac{1}{3}h(a^2 + \beta ab + b^2),$$

where β is an unknown constant. The $a = 0$ and $b = 0$ cases will work because of the coefficient of one-third. So now choose the coefficient of β to make the $a = b$ special case work. When $a = b$, the proposed volume becomes

$$V = \frac{2 + \beta}{3}hb^2.$$

Since the correct volume is hb^2, the only possibility is $\beta = 1$. Therefore

$$V = \frac{1}{3}h(a^2 + ab + b^2),$$

which you can confirm by integration.

6. Fog

a. Estimate the terminal speed of fog droplets (radius $\sim 10\,\mu m$). Use either the low- or high-Reynolds-number limit for the drag force, whichever you guess is the more likely to be valid.

Here is the low-Reynolds-number terminal velocity from the lecture notes:

$$v \sim \frac{2}{9}\frac{gr^2}{\nu}\left(\frac{\rho_{obj}}{\rho_{fl}} - 1\right).$$

Here ρ_{obj} is the density of water, which is much greater than ρ_{fl}, the density of air. So the -1 is not important. With that simplification and calling $2/9 = 1/4$,

$$v \sim \frac{1}{4} \times \frac{10\,\text{m}\,\text{s}^{-2} \times 10^{-10}\,\text{m}^2}{10^{-5}\,\text{m}^2\,\text{s}^{-1}} \times 1000 \sim 2\,\text{cm}\,\text{s}^{-1}.$$

b. Use the speed to estimate the Reynolds number and check that you used the correct limit for the drag force. If not, try the other limit!

The Reynolds number is roughly

$$\text{Re} \sim \frac{10^{-5}\,\text{m} \times 2 \cdot 10^{-2}\,\text{m}\,\text{s}^{-1}}{10^{-5}\,\text{m}^2\,\text{s}^{-1}} \sim 0.02.$$

It is much less than 1, so the original assumption of low-Reynolds-number flow is okay.

c. Fog is a low-lying cloud. How long would fog droplets take to fall 1 km (the height of a typical cloud)? What is the everyday effect of this settling time?

> At $2\,\mathrm{cm\,s^{-1}}$, it takes $5 \cdot 10^4\,\mathrm{s}$ to fall 1 km. A day is roughly $10^5\,\mathrm{s}$, so the fall time is about one-half of a day. The everyday consequence is that fog settles overnight: You go to sleep with a pea-soup fog, and by the time you wake up, it's mostly settled onto the ground and maybe evaporated as the morning sun warms the ground.

7. **Tube flow**

   In this problem you study fluid flow through a narrow tube. The quantity to predict is Q, the volume flow rate (volume per time). This rate depends on:

l	the length of the tube
Δp	the pressure difference between the tube ends
r	the radius of the tube
ρ	the density of the fluid
ν	the kinematic viscosity of the fluid

   a. Find three independent dimensionless groups G_1, G_2, and G_3 from these six variables. *Hint 1:* One physically reasonable group is $G_2 = r/l$. *Hint 2:* Put Q in G_1 only! Then write the general form

   $$G_1 = f(G_2, G_3).$$

   [There are lots of choices for G_1 and G_3.]

   > Take $G_2 = r/l$ as a start. I'll put Q in G_1 by trying to construct another volume per time. The simplest is $r\nu$. So
   >
   > $$G_1 \equiv \frac{Q}{r\nu}.$$
   >
   > The third group has to contain Δp, otherwise it has nowhere to live, and therefore also has to contain ρ, the only other quantity that has mass among its dimensions. The dimensions of $\Delta p/\rho$ are $\mathrm{L^2 T^{-2}}$. The only way to get rid of the $\mathrm{T^{-2}}$ but without using Q is to divide by ν^2. The dimensions of $\Delta p/\rho\nu^2$ are $\mathrm{L^{-2}}$. So a dimensionless group is
   >
   > $$G_3 \equiv \Delta p \, \frac{r^2}{\rho\nu^2}.$$
   >
   > The general form is
   >
   > $$\frac{Q}{r\nu} = f\left(\frac{r}{l}, \Delta p \, \frac{r^2}{\rho\nu^2}\right).$$

b. Now imagine that the tube is very long and thin ($l \gg r$) and that the radius or flow speed are small enough to make the Reynolds number low. Then you can deduce the form of f using proportional reasoning.

You might think about these proportionalities:

1. How should Q depend on Δp? For example, if you double the pressure difference, what should happen to the flow rate?

> In this oozing-flow limit, the drag is directly from viscosity. And viscous drag is proportional to velocity. Doubling the pressure difference should double the viscous drag that can be overcome, which should double the speed, which should double the flow rate. So $Q \propto \Delta p$.

2. How should Q depend on l? For example, if you keep the pressure difference the same but double the tube length, what happens to Q? Or if you double Δp and double l, what happens to Q?

> Doubling Δp and doubling l is just like connecting two tubes in sequence, each with the original Δp and l. The flow rate from one tube smoothly feeds the second tube. So the flow rate of the combined system, which has the doubled Δp and doubled l, is the same as the flow rate of the original system.
>
> So if $\Delta p / l$ does not change, neither should Q, which means that $Q \propto \Delta p / l$.

Figure out the form of f to satisfy all your proportionality requirements.

If you get stuck going forward, instead work backward from the correct result. Look up Poiseuille flow, and use this result to deduce the preceding proportionalities; and then give reasons for why they are that way.

> Let's guess that $f(G_2, G_3)$ is $G_2^n G_3^m$. To get $Q \propto \Delta p$, the exponent m must be 1. To get $Q \propto \Delta p / l$, the exponent n must also be 1. So
> $$\frac{Q}{rv} \sim \frac{r}{l} \frac{\Delta p r^2}{\rho v^2},$$
> or
> $$Q \sim \frac{\Delta p}{l} \frac{r^4}{\rho v}.$$

c. *[optional]*
The dimensional analysis in the preceding parts does not tell you the dimensionless constant. Use a syringe and needle to estimate the constant. Compare your constant with the value of $\pi/8$ that comes from solving the equations of fluid mechanics honestly.

From the preceding parts, the dimensionless constant is

$$C = Q \frac{l}{\Delta p} \frac{\rho v}{r^4}.$$

I used a 27-gauge needle, which has $r \sim 0.1\,\text{mm}$ and $l \sim 3\,\text{cm}$, attached to a 3cc syringe. I filled the syringe with 1.5cc of water, and placed it upside down on a kitchen scale so that the plunger was touching the scale and the needle was in the air. Then I pushed it downward so that the scale read 200 g and timed how long it took to squeeze out 1cc. It took roughly 30 s.

A 200 g scale reading means a force of $mg \sim 2\,\text{N}$. It is distributed over the syringe barrel, which has diameter 1 cm, so the pressure is

$$\Delta p \sim \frac{2\,\text{N}}{\frac{\pi}{4} \times 10^{-4}\,\text{m}^2} \sim 2.5 \cdot 10^4\,\text{Pa}.$$
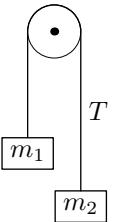
Putting in all the numbers:

$$C \sim \frac{10^{-6}\,\text{m}^3}{30\,\text{s}} \times \frac{3 \cdot 10^{-2}\,\text{m}}{2.5 \cdot 10^4\,\text{Pa}} \times \frac{10^{-3}\,\text{kg}\,\text{m}^{-1}\,\text{s}^{-1}}{10^{-16}\,\text{m}^4} \sim 0.4.$$

The true value is $\pi/8 \approx 0.39$!

8. **Atwood machine: Tension in the string**

   Here is the Atwood machine from lecture. The string and pulley are massless and frictionless. We used dimensional analysis and special cases to guess the acceleration of either mass. With the right choice of sign,

   $$\frac{a}{g} = \frac{m_1 - m_2}{m_1 + m_2}.$$

   In this problem you guess the tension in the string.

   a. The tension T, like the acceleration, depends on m_1, m_2, and g. Explain why these four variables result in two independent dimensionless groups.

   > There are only two independent dimensions, for example M and LT^{-2}. So four variables result in two groups.

   b. Choose two suitable independent dimensionless groups so that you can write an equation for the tension in this form:

      dimensionless group containing $T = f$(dimensionless group without T).

      The next part will be easier if you use a lot of symmetry in choosing the groups.

      > For one group, use the same one as in lecture when we found the acceleration:
      >
      > $$G_2 \equiv \frac{m_1 - m_2}{m_1 + m_2}.$$
      >
      > The other group has to contain the tension T. It has dimensions of force, as does $m_1 g$. So we could use $T/m_1 g$. But that choice is not symmetric under interchange of m_1 and m_2. The other choice $T/m_2 g$ has the same problem. But

$$G_1 = \frac{T}{(m_1 + m_2)g}$$

is symmetric.

So,

$$\frac{T}{(m_1 + m_2)g} = f\left(\frac{m_1 - m_2}{m_1 + m_2}\right).$$

**c.** Use special cases to guess f, and sketch f.

The special cases that helped in guessing the acceleration are $m_1 = 0$, $m_2 = 0$, and $m_1 = m_2$. So use them again.

When one mass is zero, the other mass free falls, so $T = 0$, which means $f(-1) = f(1) = 0$. When $m_1 = m_2$, the system does not accelerate, so the force from the string tension balances the weight, so $T = m_1 g$ and $T = m_2 g$. Therefore $f(0) = 1/2$.

A simple function that passes through these points is

$$f(x) = \frac{1}{2}(1 - x^2).$$

Here is a sketch.
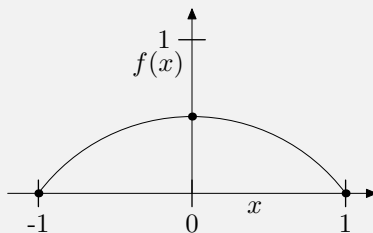
The tension is therefore

$$T = \frac{1}{2}(m_1 + m_2)g\left(1 - \left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2\right) = \frac{2m_1 m_2}{m_1 + m_2}g.$$

This form has an intuitive and compact representation:

$$T = 2(m_1 \| m_2)g,$$

where $a\|b$ means the parallel combination of a and b as if they were resistances.

d. Solve for T using the usual methods from introductory physics (8.01); then compare that answer with your answer from the preceding part.

The net downward on m_1 is $m_1 g - T$, so its downward acceleration is $g - T/m_1$. The net upward force on m_2 is $T - m_2 g$, so its upward acceleration is $T/m_2 - g$.

These accelerations are equal, so

$$g - \frac{T}{m_1} = \frac{T}{m_2} - g.$$

Collect all the g's to the left side and all the terms with T on the right side. Then

$$2g = T\left(\frac{1}{m_2} + \frac{1}{m_1}\right).$$

Therefore

$$T = \frac{2m_1 m_2}{m_1 + m_2} g,$$

which matches the result from dimensional analysis and special-cases reasoning.

Optional

9. **Plant-watering system**

The semester is over and you are going on holiday for a few weeks. But how will you water the house plants?! Design an unpowered slow-flow system to keep your plants happy.

One method is to use Poiseuille flow by choosing the pressure gradient and pipe diameter to get a slow-enough flow.

One of my plants needs a cup of water ($\sim 200\,\mathrm{cm}^3$) every day, so

$$Q \sim \frac{2 \cdot 10^{-4}\,\mathrm{m}^3}{10^5\,\mathrm{s}} \sim 2 \cdot 10^{-9}\,\mathrm{m}^3\,\mathrm{s}^{-1}.$$

From an earlier problem,

$$Q \sim \frac{\Delta p}{l} \frac{r^4}{\rho v}.$$

To make Q very tiny, the best way is to use a small pipe radius r, because r shows up with a fourth power. I'll see how well $r = 0.1\,\mathrm{mm}$ works.

Another part of the problem is how to make the pressure gradient. I'll let gravity generate the gradient by keeping the water in a tall tank of height h (with a plastic sheet as a cover to prevent evaporation) and using the hydrostatic pressure $\rho g h$ as the driving pressure. Then $\Delta p = \rho g h$ and
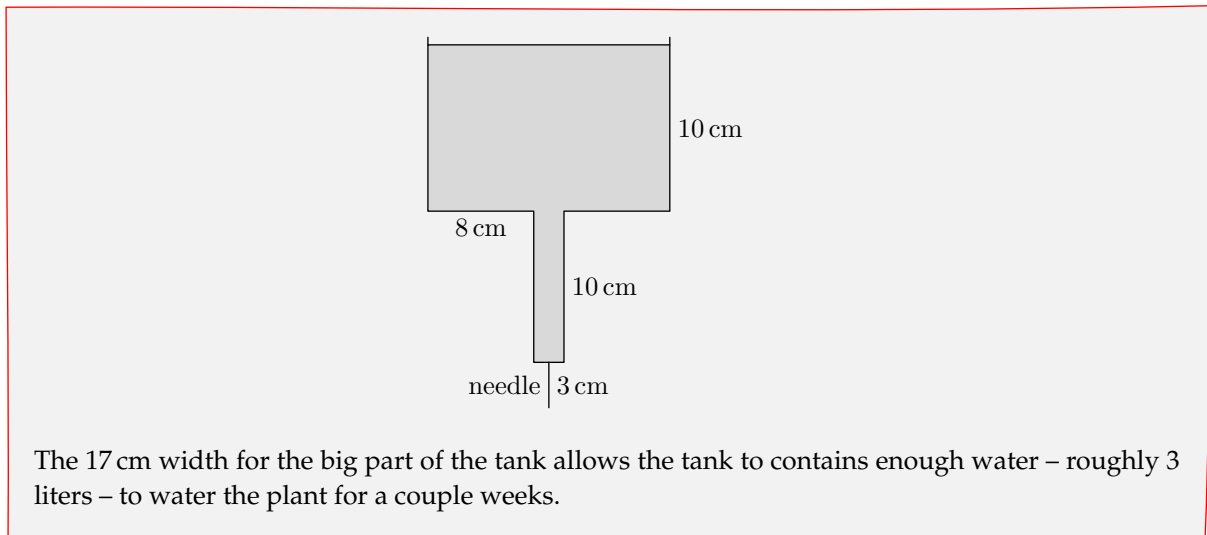
$$Q \sim \frac{gh}{l} \frac{r^4}{v} = g\frac{r^4}{v}\frac{h}{l}.$$

With $r = 0.1\,\mathrm{mm}$, the flow rate is

$$Q \sim 10\,\mathrm{m\,s^{-2}} \times \frac{10^{-16}\,\mathrm{m}^4}{10^{-6}\,\mathrm{m}^2\,\mathrm{s}^{-1}} \times \frac{h}{l} = 10^{-9}\,\mathrm{m}^3\,\mathrm{s}^{-1} \times \frac{h}{l}.$$

So $h/l \sim 2$ in order to get the desired $Q \sim 2 \cdot 10^{-9}\,\mathrm{m}^3\,\mathrm{s}^{-1}$. Actually, if I include the factor of $\pi/8$, then I need $h/l \sim 5$. One way to get $r \sim 0.1\,\mathrm{mm}$ is to use a 27-gauge needle. A typical 27-gauge needle – at least, the ones I've used for giving myself allergy treatments – has $l \sim 3\,\mathrm{cm}$. So I'll need $h \sim 15\,\mathrm{cm}$.

It's not easy to keep h fixed at 15 cm for two weeks. But if h does not vary too much, then the flow will be constant enough. I'll let h vary between 10 cm and 20 cm with this arrangement:

The 17 cm width for the big part of the tank allows the tank to contains enough water – roughly 3 liters – to water the plant for a couple weeks.

10. Dimensional analysis for circuits

a. Using Q as the dimension of charge, what are the dimensions of inductance L, capacitance C, and resistance R?

Resistance shows up in the relation between power and current: $P = I^2 R$. Since the dimensions of I are QT^{-1}, the dimensions of R are

$$[R] = \frac{[P]}{[I^2]} = \frac{ML^2T^{-3}}{Q^2T^{-2}} = Q^{-2}ML^2T^{-1}.$$

Yuk.

Capacitance shows up in $Q = CV$, so

$$[C] = \frac{Q}{[V]}.$$

Voltage is energy per charge (which is why electron–Volts are a unit of energy). So

$$[C] = Q^2M^{-1}L^{-2}T^2.$$

Inductance shows up in $V = L\, dI/dt$, so

$$[L] = \frac{[V]}{[I/t]} = \frac{Q^{-1}ML^2T^{-2}}{QT^{-2}} = Q^{-2}ML^2.$$

b. Show that the dimensions of L, C, and R contain two independent dimensions.

I can construct the dimensions of L, R, and C using $Q^{-2}ML^2$ and T.

c. In a circuit with one inductor, one capacitor, and one resistor, one dimensionless group should result from the three component values L, R, and C. What is physical interpretation of this group?

> Reasonable dimensionless group are $\sqrt{L/C}/R$ or its square L/RC^2. If I use the first choice, it is also known as the quality factor Q (nothing to do with charge), which measures how resonant a circuit is.