6.055J/2.038J (Spring 2008)

Solution set 3

Do the following warmups and problems. Due in class on ***Friday, 14 Mar 2008****.*

Open universe: *Collaboration, notes, and other sources of information are* ***encouraged.*** *However, avoid looking up answers until you solve the problem (or have tried hard). That policy helps you learn the most from the problems.*

Bring a photocopy to class on the due date*, trade it for a solution set, and figure out or ask me about any confusing points. Your work will be graded lightly: P (made a reasonable effort), D (did not make a reasonable effort), or F (did not turn in).*

Warmups

1. Explain a Unix pipeline

What does this pipeline do?

```
ls -t | head | tac
```

[Hint: If you are not familiar with Unix commands, use the `man` command on Athena or on any nearby Unix or GNU/Linux system.]

The `ls -t` lists the files and subdirectories in a directory ordered by modification time with the most recently modified files at the beginning. The `head` selects the first ten lines, which means the first ten names. The `tac` reverses the order of the lines, so the 10th-most-recently-modified file (or subdirectory) comes first, then the 9th-most-recently-modified file, etc. with the most-recently-modified file at the end of the list.

2. Symmetry for algebra

Use symmetry to find $(a-b)^3$.

The original expression is antisymmetric in a and b: The result changes sign if you swap a and b.

The expansion has third-degree terms such as a^3 or a^2b. One category of third-degree terms is like a^3 and includes a^3 and b^3. The antisymmetric combination is $a^3 - b^3$. The other category of third-degree terms is like a^2b and includes a^2b and ab^2. The antisymmetric combination is $a^2b - ab^2$.

The expansion therefore has the antisymmetric form

$$(a-b)^3 = A(a^3 - b^3) + B(a^2b - ab^2)$$

where A and B are constants to be determined.

Setting $b = 0$ shows that $A = 1$, because $(a-0)^3 = A(a^3 - 0) + B(0-0)$ or $a^3 = Aa^3$.

To find B, think about the naive expansion of $(a-b)^3$. The basic expression $a - b$ has two terms, so $(a-b)^3$ has eight terms (before collecting like terms). So the absolute values of the coefficients of each term in the form

$$A(a^3 - b^3) + B(a^2b - ab^2)$$

have to add to eight. With $A = 1$, this requirement shows that $B = \pm 3$. The choice $B = -3$ gives the correct sign for the a^2b term (which has one negative factor from the $-b$).

So

$$(a - b)^3 = (a^3 - b^3) - 3(a^2b - ab^2).$$

Problems

3. Highway vs city driving

In lecture we derived a measure of how important drag is for a car moving at speed v for a distance d:

$$\frac{E_{\text{drag}}}{E_{\text{kinetic}}} \sim \frac{\rho v^2 Ad}{m_{\text{car}} v^2}.$$

a. Show that the ratio is equivalent to the ratio

$$\frac{\text{mass of the air displaced}}{\text{mass of the car}}$$

and to the ratio

$$\frac{\rho_{\text{air}}}{\rho_{\text{car}}} \times \frac{d}{l_{\text{car}}},$$

where ρ_{car} is the density of the car (i.e. its mass divided by its volume) and l_{car} is the length of the car.

In the ratio $\rho v^2 Ad/m_{\text{car}}v^2$, the v^2 divide out leaving $\rho Ad/m_{\text{car}}$, where ρ is the air density. Since A is the cross-sectional area of the car, Ad is the volume of air that the car displaces, and ρAd is the mass of that air. So

$$\frac{E_{\text{drag}}}{E_{\text{kinetic}}} \sim \frac{\rho v^2 Ad}{m_{\text{car}} v^2} = \frac{\rho Ad}{m_{\text{car}}} = \frac{\text{mass of the air displaced}}{\text{mass of the car}}.$$

An alternative equivalence comes from writing the mass of the car as $\rho_{\text{car}} A l_{\text{car}}$. Making that substitution and dividing out the v^2 gives

$$\frac{\rho v^2 Ad}{m_{\text{car}} v^2} = \frac{\rho_{\text{air}} Ad}{\rho_{\text{car}} A l_{\text{car}}} = \frac{\rho_{\text{air}}}{\rho_{\text{car}}} \frac{d}{l_{\text{car}}}.$$

b. Make estimates for a typical car and find the distance d at which the ratio becomes significant (say, roughly 1). How does the distance compare with the distance between exits on the highway and between stop signs or stoplights on city streets?

A typical car has mass $m_{car} \sim 10^3$ kg, cross-sectional area $A \sim 2\,\mathrm{m} \times 1.5\,\mathrm{m} = 3\,\mathrm{m}^2$, and length $l_{car} \sim 4$ m. So

$$\rho_{car} \sim \frac{m_{car}}{Al_{car}} \sim \frac{10^3\,\mathrm{kg}}{3\,\mathrm{m}^2 \times 4\,\mathrm{m}} \sim 10^2\,\mathrm{kg\,m^{-3}}.$$

Since $\rho_{car}/\rho_{air} \sim 100$, the ratio

$$\frac{\rho_{air}}{\rho_{car}} \frac{d}{l_{car}}$$

becomes 1 when $d/l_{car} \sim 100$, so $d \sim 400$ m.

This distance d is significantly farther than the distance between stop signs or stoplights on city streets. In Manhattan, for example, 20 east–west blocks are one mile, giving a spacing of approximately 80 m. So air resistance is not a significant loss in city driving. Instead the loss comes from engine friction, rolling resistance, and braking.

However, the distance d is comparable to the exit spacing on urban highways. So when you drive on the highway for even a few exit distances, air resistance is a significant loss.

Interestingly, highway fuel efficiencies are higher than city fuel efficiencies, even though drag gets worse at the higher, highway speeds, and presumably engine friction and rolling resistance also get worse at higher speeds. Only one loss mechanism, braking, is less prevalent in highway than in city driving. So braking must cause a significant loss in city driving. Regenerative braking, for hybrid or electric cars, should significantly improve fuel efficiency in city driving.

4. **Symmetry for second-order systems**

This problem analyzes the frequency of maximum gain for an *LRC* circuit or, equivalently, for a damped spring–mass system. The gain of such a system is the ratio of the input amplitude to the output amplitude as a function of frequency.

If the output voltage is measured across the resistor, and you drive the circuit with a voltage oscillating at frequency ω, the gain is (in a suitable system of units):

$$G(\omega) = \frac{j\omega}{1 + j\omega/Q - \omega^2},$$

where $j = \sqrt{-1}$ and Q is quality factor, a dimensionless measure of the damping.

Do not worry if you do not know where that gain formula comes from. The purpose of this problem is not its origin, but rather using symmetry to maximize its magnitude.

a. Show that the magnitude of the gain is

$$|G(\omega)| = \frac{\omega}{\sqrt{\left(1-\omega^2\right)^2 + \omega^2/Q^2}}.$$

The magnitude of the numerator is ω (assuming positive frequency). The magnitude of the denominator is $\sqrt{|\text{real part}|^2 + |\text{imaginary part}|^2}$ so it is $\sqrt{\left(1-\omega^2\right)^2 + \omega^2/Q^2}$.

The ratio of magnitudes is $|G(\omega)|$:

$$|G(\omega)| = \frac{\omega}{\sqrt{\left(1-\omega^2\right)^2 + \omega^2/Q^2}}.$$

b. Find a variable substitution (a symmetry operation) $\omega_{\text{new}} = f(\omega)$ that turns $|G(\omega)|$ into $|H(\omega_{\text{new}})|$ such that G and H are the same function (i.e. they have the same structure but with ω in G replaced by ω_{new} in H).

When maximizing a parabolic function such as $y = x(6-x)$, the symmetry is reflection about the line $x = 3$. In symbols, the transformation is $x_{\text{new}} = 6 - x$.

Let's transfer a few lessons from the parabola example to the problem of maximizing the gain. In the parabola example, the symmetry is a reflection about an interesting point (there, the point halfway between the two roots $x = 0$ and $x = 6$). Analogously, an interesting frequency is $\omega = 1$ because it makes the real part of the denominator in $G(\omega)$ go to zero, and making the real part go to zero helps minimize the denominator.

Therefore reflecting about $\omega = 1$ is worth trying, perhaps $\omega_{\text{new}} = 1 - \omega$. For frequencies, however, differences are not as important as ratios. For example, a musical octave is a factor of 2 in frequency, rather than a difference. So reflect in a multiplicative way: $\omega_{\text{new}} = w^{-1}$.

This transformation works either in $G(\omega)$ or in the magnitude $|G(\omega)|$. It's slightly easier in $G(\omega)$:

$$G(\omega) = \frac{j\omega}{1 + j\omega/Q - \omega^2} \mapsto H(\omega_{\text{new}}) = \frac{j/\omega_{\text{new}}}{1 + j/Q\omega_{\text{new}} - 1/\omega_{\text{new}}^2}.$$

Multiply numerator and denominator by ω_{new}^2:

$$H(\omega_{\text{new}}) = \frac{j\omega_{\text{new}}}{\omega_{\text{new}}^2 + j\omega_{\text{new}}/Q - 1},$$

which is the same function as $G(\omega)$, except for negating the real part in the denominator. Negating the real part in the denominator doesn't affect the magnitude of the denominator, so $|H(\omega_{\text{new}})|$ has the same form as $|G(\omega)|$.

c. Use the form of that symmetry operation to maximize $|G(\omega)|$ without using calculus.

Since $\omega_{\text{new}} = 1/\omega$, the maximum value of ω_{new} will be ω_{max}^{-1}. That's one equation.

Since the two magnitudes $|G(\omega)|$ and $|H(\omega_{\text{new}})|$ are the same function, the maximum value of ω_{new} is also the maximum value of ω. That's the second equation.

Together they produce $\omega = \omega_{\text{new}} = 1$ (ignoring the negative-frequency solution $\omega = -1$). At that frequency, $|G(\omega)|$ is Q. For the electrical and mechanical engineers: The quality factor Q is also the gain at resonance.

d. *[Optional, for masochists!]* Maximize $|(G\omega)|$ using calculus.

A direct differentiation of $|G(\omega)|$ is too awful for words, and I cannot make myself do it. Instead I'll massage the expression until differentiating is not horrible or maybe not even needed.

First, put the ω from the numerator into the denominator by multiplying numerator and denominator by $1/\omega$:

$$|G(\omega)| = \frac{\omega}{\sqrt{\left(1-\omega^2\right)^2 + \omega^2/Q^2}} = \frac{1}{\sqrt{\left(\omega^{-1}-\omega\right)^2 + 1/Q^2}}.$$

Second, find the extremum of an equivalent, simpler expression. Maximizing $1/\sqrt{f(\omega)}$ is equivalent to maximizing $1/f(\omega)$. And maximizing $1/f(\omega)$ is equivalent to minimizing $f(\omega)$. So I'll find the extremum of

$$\left(\omega^{-1}-\omega\right)^2 + 1/Q^2.$$

Furthermore, the $1/Q^2$ doesn't affect the location of the extremum, so instead I minimize $\left(\omega^{-1} - \omega\right)^2$. Even better, the squaring does not affect the location of the extremum, so I minimize the absolute value of $\omega^{-1} - \omega$. Its absolute value can never fall below zero, and it equals zero when $\omega = 1$. So $\omega = 1$ is the location of the maximum of $|G(\omega)|$. No need for differentiation!

5. Gravity on the moon

In this problem you use a scaling argument to estimate the strength of gravity on the surface of the moon.

a. Assume that a planet is a uniform sphere. What is the proportionality between the gravitational acceleration g at the surface of a planet and the planet's radius R and density ρ?

The force of gravity on an object of mass m is $F = mg$. By Newton's law of gravitation, it is also $F = GMm/R^2$, where M is the mass of the planet and G is Newton's constant of gravitation. Therefore the gravitational acceleration is $g = GM/R^2$. Since $M = \rho(4\pi/3)R^3$, the gravitational acceleration is

$$g = \frac{G\rho(4\pi/3)R^3}{R^2} \propto \rho R.$$

In the last step, G vanished and the equals sign got replaced by a proportionality, which is okay since G is the same for all planets in the universe.

b. Write the ratio $g_{\text{moon}}/g_{\text{earth}}$ as a product of dimensionless factors as in the analysis of the fuel efficiency of planes.

Using the proportionality, the ratio of gravities is

$$\frac{g_{\text{moon}}}{g_{\text{earth}}} = \frac{\rho_{\text{moon}}}{\rho_{\text{earth}}} \frac{R_{\text{moon}}}{R_{\text{earth}}}.$$

The factors are ratios of densities or radii, so they are dimensionless.

c. Estimate those factors and estimate the ratio g_{moon}/g_{earth}, then estimate g_{moon}. [Hint: To estimate the radius of the moon, whose angular size you can estimate by looking at it, you might find it useful to know that the moon is $4 \cdot 10^8$ m distant from the earth.]

I'll first assume that earth and moon rock are the same. So $\rho_{moon}/\rho_{earth} \sim 1$.

The earth's radius is worth memorizing once you've derived it. Here's one way to derive it. The distance from Boston to San Francisco is about 3000 miles, and the cities are separated by three time zones. So the sun 'travels' about 1000 miles per time zone (per hour). Since one day has 24 time zones, the sun's travel around the earth is about 24,000 miles. That value is the circumference $2\pi R_{earth}$, so $R_{earth} \sim 4 \cdot 10^3$ mi (since $\pi \sim 3$) or $6.4 \cdot 10^3$ km. This estimate neglects a trigonometric factor due Boston not being on the equator, but it makes other errors, and they cancel (surprise!): The true value of the mean radius is 6373 km.

The moon's radius needs a different analysis. I can just cover the moon with my index finger at arm's length. So the moon subtends an angle

$$\theta \sim \frac{\text{width of my finger}}{\text{my arm length}} \sim \frac{1\,\text{cm}}{1\,\text{m}} \sim 0.01.$$

So the *diameter* of the moon is roughly θd, where $d \sim 4 \cdot 10^8$ m is the distance to the moon, and the radius is therefore $R_{moon} \sim 2 \cdot 10^6$ m. If the moon is hidden, you can (carefully!) use the sun instead because it has the same angular size as the moon – which is the explanation for total solar eclipses.

The density and radii factors produce

$$\frac{g_{moon}}{g_{earth}} = \underbrace{1}_{\text{densities}} \times \underbrace{\frac{1}{3}}_{\text{radii}} = \frac{1}{3}.$$

So $g_{moon} \sim 3\,\text{m}\,\text{s}^{-2}$.

d. Look up g_{moon} and compare the value to your estimate, venturing an explanation for any discrepancy.

The true value is $g_{moon} \sim 1.6\,\text{m}\,\text{s}^{-2}$. So the estimate is too high by a factor of 2. The radii estimates are fairly accurate, so the equal-density assumption must be significantly wrong. So the moon's density is much less than the earth's. The actual values are $\rho_{moon} \sim 3.4\,\text{g}\,\text{cm}^{-3}$ and $\rho_{earth} \sim 5.5\,\text{g}\,\text{cm}^{-3}$. However, ρ_{moon} is comparable to the density of rock in the earth's crust. Perhaps the moon was once part of the earth's crust, which is still a viable theory of the moon's origin.

6. Checking plane fuel-efficiency calculation

This problem offers two more methods to estimate the fuel efficiency of a plane.

a. Use the cost of a plane ticket to estimate the fuel efficiency of a 747, in passenger–miles per gallon.

A roundtrip ticket from New York to San Francisco costs roughly $400. The journey is about 2500 miles each way, so a 5000-mile journey costs about $500 (rounding up the $400 to make the math easier). That's about 10 cents/mile. Perhaps one-half of that cost is fuel. [Although the service – in the air, on the phone, and at the counter – is so lousy due to understaffing that perhaps two-thirds of the cost being fuel would be a better estimate!] At 5 cents/mile for fuel, and at $3/gallon for fuel, the fuel efficiency is 60 passenger–miles per gallon.

b. According to Wikipedia, a 747-400 can hold up to $2 \cdot 10^5 \ell$ of fuel for a maximum range of $1.3 \cdot 10^4$ km. Use that information to estimate the fuel efficiency of the 747, in passenger–miles per gallon.

The 747 can hold about 400 people, so the fuel efficiency is

$$\frac{400\,\text{passengers} \times 1.3 \cdot 10^4\,\text{km}}{2 \cdot 10^5 \ell} \times \frac{1\,\text{mile}}{1.6\,\text{km}} \times \frac{4\ell}{1\,\text{gallon}} \sim 65\,\text{passenger–miles per gallon.}$$

This estimate is amazingly close to the estimate from using the ticket price!

How do these values compare with the rough result from lecture, that the fuel efficiency is comparable to the fuel efficiency of a car?

The fuel efficiency of a medium-sized car (holding one person, which is typical for California) is roughly 30 passenger–miles per gallon. So both fuel-efficiency estimates in this problem give a fuel efficiency that is a factor of 2 higher than the result from lecture – not too bad considering how much we neglected (drag coefficient and lift being the main ones) when we estimated the efficiency.

7. Invent your own problem

Invent your own problem whose solution might use symmetry, proportional reasoning, or a Unix pipeline.

Optional

8. Design a Unix pipeline

Make a pipeline that prints the ten most common words in the input stream, along with how many times each word occurs. They should be printed in order from the the most frequent to the less frequent words. [Hint: First **tr**anslate any non-alphabetic character into a newline. Useful utilities include `tr` and `uniq`.]

Divide and conquer! The first step is to get rid of all the non-alphabetic characters and turn them into newlines. Then get rid of the empty lines, which occur either from empty lines in the original text or when consecutive non-alphabetic characters get turned into newlines. Then we'll have the words from the file, one word per line. This piece of the pipeline is

```
tr -cs 'a-zA-Z' '\n'
```

The `-c` option says that the list of characters is to be inverted (complemented). So `tr` will translate all characters except for the upper- and lowercase alphabetic characters `a-z` and `A-Z`. The backslash-`n` is the Unix syntax for the newline character. The `-s` option tells `tr` to squeeze repeated translated characters into one copy of that character; therefore repeated newlines get turned into one newline, which gets rid of the empty lines.

To count the words, sort them and run `uniq`. `uniq` looks only at adjacent lines, which is why the word list needs to be sorted. In the simplest invocation, `uniq` print the first line from a series of duplicate lines. For example, feeding this input to `uniq`

```
the
the
the
how
the
how
how
```

produces

```
the
how
the
how
```

Giving `uniq` the `-c` switch tells it instead to count the duplicates. The same input to `uniq -c` produces

```
3 the
1 how
1 the
2 how
```

The pipeline so far is

```
tr -cs 'a-zA-Z' '\n' | sort | uniq -c
```

I want the top ten words, so I reverse sort the list numerically (so that the largest count is at the top) with `sort -nr`, then select the top ten lines with `head`.

The full pipeline is

```
tr -cs 'a-zA-Z' '\n' | sort | uniq -c | sort -nr | head
```

As a test, here is the result of applying that pipeline to an old email message about misconceptions about gravity on the moon. The full command is:

```
tr -cs 'a-zA-Z' '\n' < email.txt | sort | uniq -c | sort -nr | head
```

It produces this word-frequency list:

```
149 the
 87 it
 65 is
 53 to
 52 Moon
 50 of
 44 will
 43 float
 33 on
 33 away
```