Approximating logarithms using musical intervals

Semitones	*Interval*	*Ratio*	*Exact Value*
2	M2	$9/8$	1.122
3	m3	$6/5$	1.1885
4	M3	$5/4$	1.259
5	P4	$4/3$	1.3335
6	d5	$\sqrt{2}$	1.4125
7	P5	$3/2$	1.496
8	$\mathrm{m6} = \mathrm{P8} - \mathrm{M3}$	$8/5$	1.585
9	$\mathrm{M6} = \mathrm{P8} - \mathrm{m3}$	$5/3$	1.679
10	$\mathrm{P5} + \mathrm{m3}$	$9/5$	1.7783
	$2 \cdot \mathrm{P4}$	$16/9$	1.7783
11		$17/9$	1.8836
12	P8	2	1.9953
17.4		e	2.718
19	$\mathrm{P8} + \mathrm{P5}$	3	2.9854
24	$2 \cdot \mathrm{P8}$	4	3.981
28	$2 \cdot \mathrm{P8} + \mathrm{M3}$	5	5.012
31	$2 \cdot \mathrm{P8} + \mathrm{P5}$	6	5.9566
34	$3 \cdot \mathrm{P8} - \mathrm{M2}$	$\frac{64}{9} \approx 7$	7.080
36	$3 \cdot \mathrm{P8}$	8	7.943
38	$2 \cdot (\mathrm{P8} + \mathrm{P5})$	9	8.913
40	$3 \cdot \mathrm{P8} + \mathrm{M3}$	10	10.

KEY		
Symbol	*Interval*	*Notes*
M2	Major 2nd	C–D
m3	Minor 3rd	C–E♭
M3	Major 3rd	C–E
P4	Perfect 4th	C–F
d5	Diminished 5th	C–G♭
P5	Perfect 5th	C–G
m6	Minor 6th	C–A♭
M6	Major 6th	C–A

The starting point is $2^{10} \approx 10^3$, or $2^{1/12} \approx 10^{1/40}$. By chance $2^{1/12}$ is the semitone frequency ratio on the equal-tempered scale. Since we know what Pythagorean ratios the equal-tempered intervals are supposed to approximate, we can approximate logarithms to the base $2^{1/12}$, and thereby approximate logarithms to the base $10^{1/40}$, which gives us twice the number of decibels. The ratio column indicates the ratios for perfect Pythagorean intervals, and the exact value column shows $10^{\text{semitones}/40}$, to show the accuracy of the method. Note that 10 semitones has two possible breakdowns into intervals, as $\mathrm{P5} + \mathrm{m3}$ or $2 \cdot \mathrm{P4}$. The second is much more accurate, because in the equal-tempered scale, the perfect intervals come out almost exactly right, at the cost of some error in the major and minor intervals.

To use the table to compute $\log_{10} x$, find x as a product of ratios, add the number of semitones for the ratios, and divide by 40 (divide by 2 to get dB). To calculate 10^x, multiply x by 40, find that value in the semitones column, and read off the corresponding ratio. From a few basic Pythagorean ratios and number of semitones, most of the table is easy to figure out. The most important to remember one is the fifth: 7 semitones corresponds to $3/2$. For example, from the fifth we can compute the frequency ratio for a fourth (5 semitones). The two intervals together make an octave, so the product of their frequency ratios is 2. This means 5 semitones corresponds to $4/3$. Many other entries can be worked out similarly.

Some examples (arrows point from the real to the log world):

$$2 \to 1\,\text{octave} = 12 \text{ semitones} = 6\,\text{dB} = 0.3\,\text{decades}.$$

$$\left(\frac{4}{3}\right)^{10} \to 10 \cdot \mathrm{P4} = 50 \text{ semitones} = 40\,\text{semitones} + 2 \cdot \mathrm{P4} \leftarrow 10 \cdot \frac{16}{9} = 17.78 \text{ (exact 17.76)}.$$

$$5 = \frac{5}{4} \cdot 2 \cdot 2 \to \mathrm{M3} + 2 \cdot \mathrm{P8} = 28\,\text{semitones} = \frac{28}{40} \text{ or } 0.7\,\text{decades } (14\,\text{dB}).$$

$$3^{10} \to 10 \cdot (\mathrm{P8} + \mathrm{P5}) = 190\,\text{semitones} = 200 - 2 \cdot \mathrm{P4} \leftarrow 10^{200/40} \cdot \frac{9}{16} = 56250 \text{ (exact 59049)}.$$

$$e^{10} \to 10 \cdot 17.4\,\text{semitones} = 174\,\text{semitones} = 160 + 12 + 2\,\text{semitones} \leftarrow 10^4 \cdot 2 \cdot \frac{9}{8} = 22500.$$

(This method is due to the statistician I. J. Good, who credits his father.)