Recitation 8. 6.003.

10-7-11.

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Convolution + Frequency Response.

Convolution:

Idea: Already knew that we could represent a system by the "system function", $H(s)$, $H(z)$., and solve block diagrams. Can also work directly w/ time domain $h[n]$, $h(t)$.

These are inverse transforms of $H(z)$, $H(s)$, or simply unit impulse response.

Derivation:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k].$$

$$\Rightarrow \boxed{y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k].}$$ by lti

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau.$$

$$\Rightarrow \boxed{y[n] = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau.}$$

Example:

$$x[n] \longrightarrow \boxed{h_1[n]} \longrightarrow \boxed{h_2[n]} \longrightarrow y[n].$$

$x[n] = \delta[n] - a\,\delta[n-1].$

$h_1[n] = \sin(8n)\,u[n].$ What's $y[n]$?

$h_2[n] = a^n u[n].$ same. $|a|<1$

Ex (cont.):

$$x[n] \longrightarrow \boxed{h_1} \xrightarrow{w[n]} \boxed{h_2} \longrightarrow y[n].$$

$$w[n] = \sum_{k=-\infty}^{\infty} \left(\delta[k] \mp a\,\delta[k-1]\right) \sin(8(n-k))\, u[n-k].$$

$$= \sin(8n)u[n] \mp \sin(8(n-1))\,u[n-1].$$

$$y[n] = \sum \quad \cdots$$

A better way ...

This is an equivalent system:

$$x[n] \longrightarrow \boxed{h_2} \longrightarrow \boxed{h_1} \longrightarrow y[n].$$

has to be. (if we had written it as $H_1(z), H_2(z)$, then no problem).

Convolution: ~~commutes~~ associative + cummutative.

$$(x * h_1 * h_2)[n] = (x * (h_1 * h_2))[n]$$

$$= (x * (h_2 * h_1))[n].$$

for commutative.

$$(x_1 * x_2)[n] = \sum_{k=-\infty}^{\infty} x_1[k]\, x_2[n-k]$$

$$= \sum_{r=-\infty}^{\infty} x_1[n-r]\, x_2[r]$$

$r = n-k$.

$$= (x_2 * x_1)[n].$$

associative easy too. Has to work out.

Note: I could even get the same answer doing.

$$h_2[n] \to \boxed{x[n]} \to \boxed{h[n]} \to y[n].$$

Back to example:

$$(x * h_2)[n] = a^n u[n] - a \cdot a^{n-1} u[n-1].$$

$$= a^n \left(u[n] - u[n-1]\right) = a^n \delta[n] = \delta[n].$$

$$y[n] = (\delta * h)[n] = \sin(8n)\, u[n].$$

Due to "sifting" property of $\delta[n]$, that one was easier in time domain.

Frequency response:

Key observation:

$$e^{st} \longrightarrow \boxed{h(t)} \longrightarrow H(s)e^{st}$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau = e^{st} \int_{-\infty}^{\infty} h(\tau) e^{-s\tau}\, d\tau = H(s)\, e^{st}$$

for DT:

$$z^n \rightarrow \boxed{h[n]} \longrightarrow \;?$$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, z^{n-k} = z^n \sum_{k=-\infty}^{\infty} h[k] z^{-k} = H(z) z^n.$$

z^n are eigenfunctions of DT systems.

For freq. response, think on imaginary axis. $s = j\omega$.

$$e^{j\omega t} \rightarrow H(j\omega)\, e^{j\omega t}.$$

or on the unit circle: $z = e^{j\Omega}$.

$$e^{j\Omega n} \rightarrow H(j\Omega)\, e^{j\Omega n}.$$

Q: Plot the frequency response of $H(s) = \dfrac{1}{(s+j)(s-j)}$.

For a particular $s = s_0$, (say $s = j\omega_0$) multiplication by $H(s)$

is simply multiplication by a complex number.

write in polar form:

$$H(s) = |H(s)|\, e^{j\angle H(s)}$$

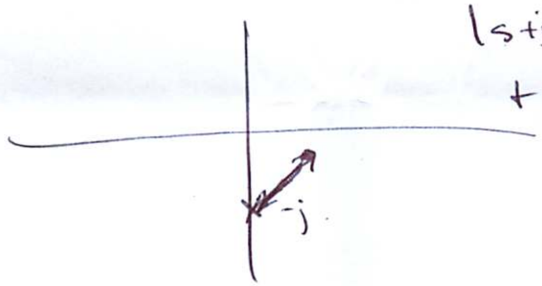
Graphical intuition w/ vector diagrams.

$$H(s) = \frac{1}{s+j}. \qquad |H(s)| = \frac{1}{|s+j|}.$$

$s+j$ is a vector in the complex plane.

$|s+j|$ is it's length.

+ $\angle s+j$ is the angle.

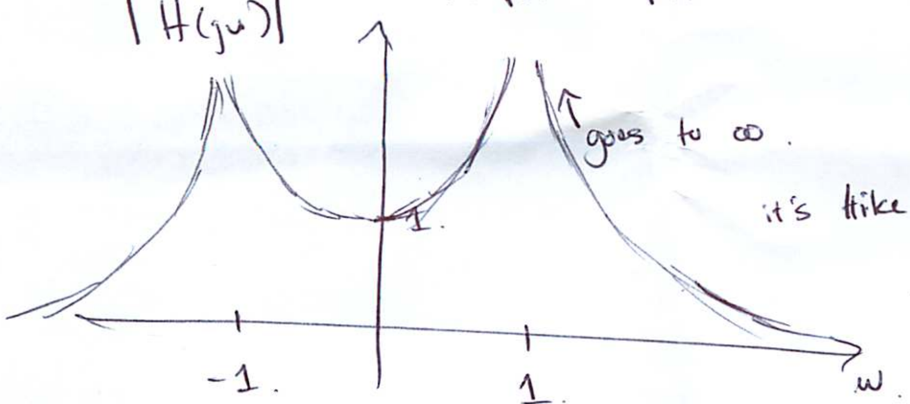


(better w/ $s-j$).

$$\angle H(s) = -\angle(s+j).$$

and $$|H(s)| = \frac{|K|\,|(s-z_0)|\,|(s-z_1)|\cdots}{|s-p_0|\,|s-p_1|\cdots}$$

$$\angle H(s) = \angle K + \sum_i \angle s-z_i - \sum_i \angle s-p_i.$$



it's like a tent pole!

does it make sense? (mass spring demo from lecture w/o the damping.