Today: Laplace & Z transforms.

6.003 Recitation 7
9.30.
Russ Tedrake.

Review:

At the point where you might feel overloaded w/ machinery.

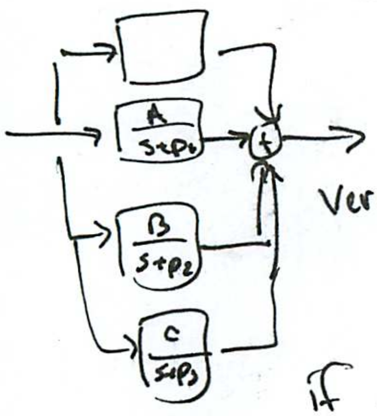
Let's take a minute to remember why we want this stuff

Most important thing we've learned:

operators take diff. eq. into polynomial form.

LCCDE → rational polynomial impulse response.

→ can rearrange (partial fraction expansion) into a sum of single pole responses!!



very hard to do in time domain. easy w/ ~~polynom~~ system functions.

if we know time domain response for single pole system, then we can write response for any.

replaced R with z^{-1}, A with $\frac{1}{s}$.

system function is $H(z) = \frac{Y(z)}{X(z)}$ or $H(s) = \frac{Y(s)}{X(s)}$.

~~combining systems~~

What if I want to know system response to arbitrary input. $x(t)$? or $x[n]$?

Turns out I can represent any function (w/ bounded integral) $x(t)$ $x[n]$ w/ $X(s)$ or $X(z)$

⇒ Laplace transform / Z transform $Y(z) = H(z) \cdot X(z)$.

One more reason you might want to go from time domain to ~~polynomials~~. z/s.

What if I describe a system in terms of it's impulse response? $h[n]$, $h(t)$.

~~You~~ (Imagine running an experiment).

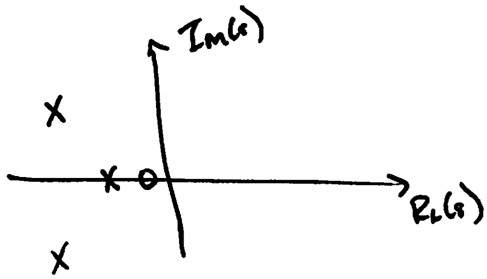
You have to build system function. $\Rightarrow$ diff. eq.

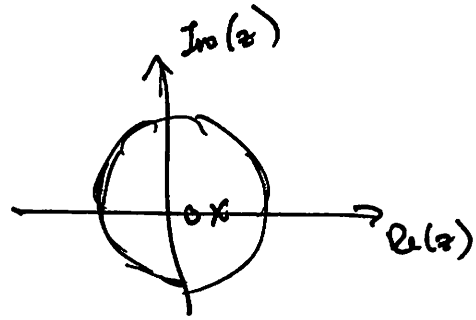
But entire motivation is that we work well w/ polynomials, and can decompose rational polys into sum of poles response.

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}. \qquad H(s) = \int_{-\infty}^{\infty} h(t) e^{-st} dt.$$

rember: not necessarily defined for all z / s.

Q: If I give you poles + zeros of a system function.





which ~~what~~ can you tell me about the system?

1) system is stable / unstable? A: yes, if we assume causal.

2) system function. A: only up to a scaling.

$$H(z) = K \cdot \frac{(z - z_0)}{(z - p_0)}.$$

Regions of convergence (R.O.C.)

Poles + zero's aren't enough for inverse transforms.

Two different signals can have same poles.

E.G. $e^{pt}u(t) \leftarrow$ right-sided

and

$-e^{pt}u(-t) \leftarrow$ left-sided.

Can see R.O.C. from transform eqs.

$$H(s) = \int_{-\infty}^{\infty} h(t)\, e^{-st}\, dt.$$

for $h(t) = e^{pt}u(t)$, for what values of s does integral converge?

$$= \int_{-\infty}^{\infty} e^{(p-s)t} u(t)\, dt = \int_{0}^{\infty} e^{(p-s)t}\, dt.$$

converges for $s > p$. (strictly greater)

poles always limit.

For C.T., vertical strips.

For DT, circles:

$$H(z) = \sum_{-\infty}^{\infty} h[n]\, z^{-n}$$

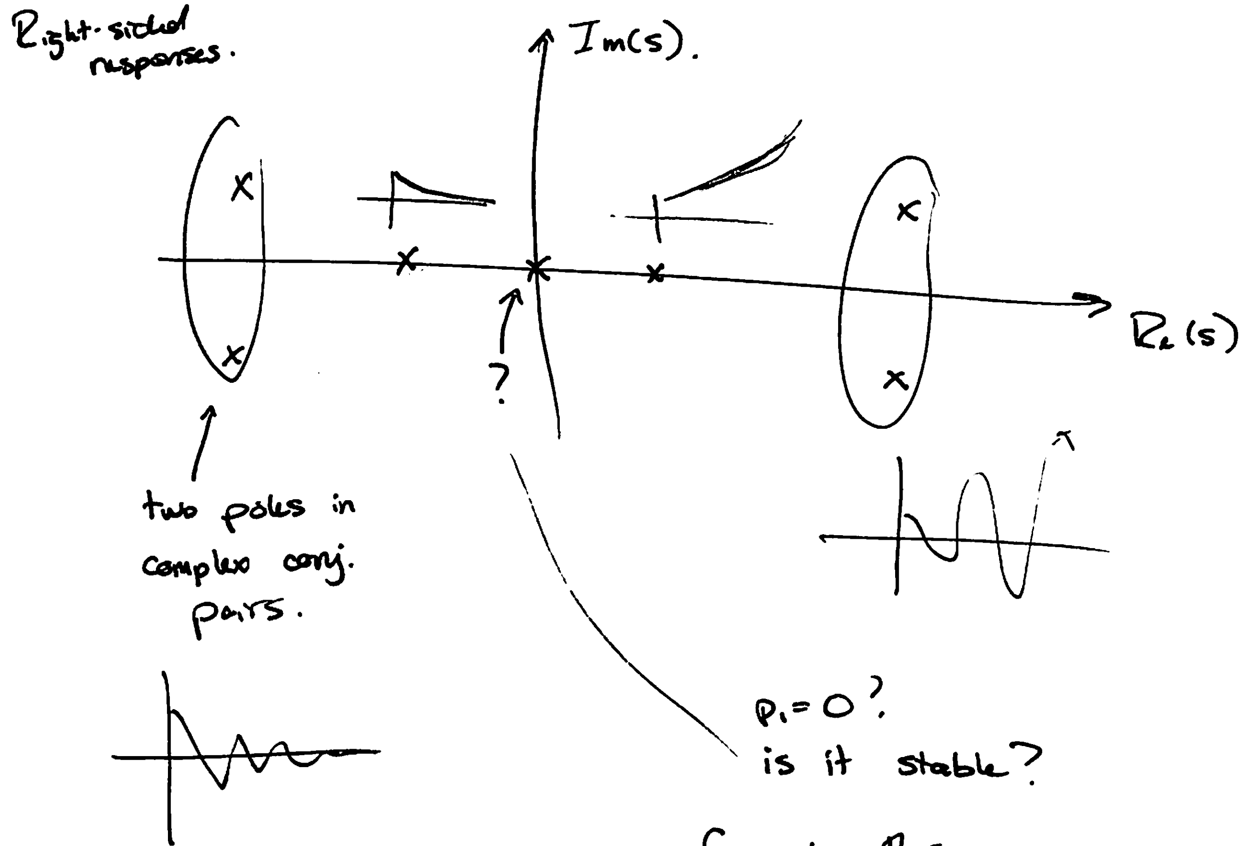
for single pole.

$$= \sum_{-\infty}^{\infty} p^n u[n]\, z^{-n} = \sum_{0}^{\infty} \left(\frac{p}{z}\right)^n. \quad |z| > |p|.$$

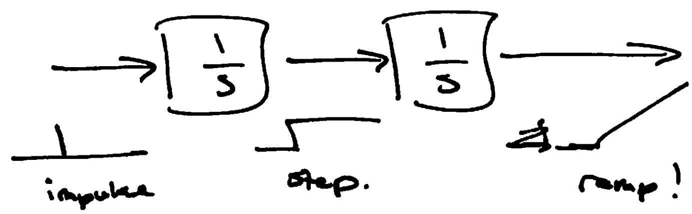
Map of responses for a single pole system.

$$H(s) = \frac{1}{s-p_1} \quad \xleftrightarrow{\text{Right-sided}} \quad h(t) = e^{p_1 t}u(t).$$

$$\xleftrightarrow{\text{Left-sided}} \quad h(t) = -e^{p_1 t}u(-t).$$

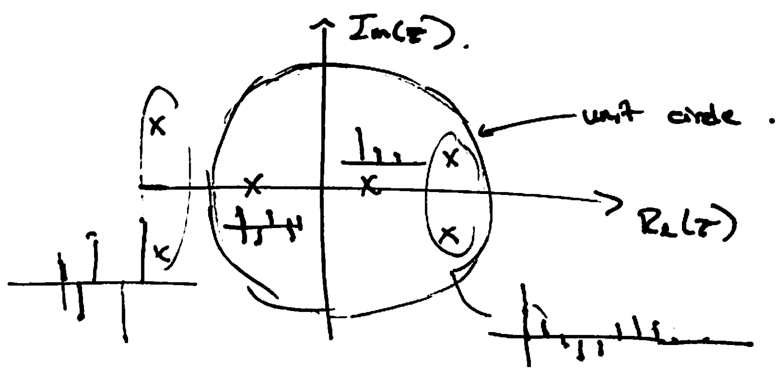


Consider this.

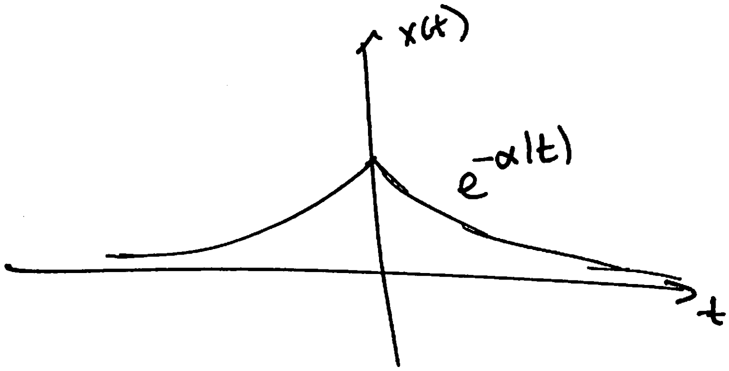


$$H(z) = \frac{1}{z-p_1} \quad \xleftrightarrow{\text{R.S.}} \quad h[n] = (p_1)^n u[n].$$

$$\xleftrightarrow{\text{L.S.}} \quad h[n] = -(p_1)^n u[-n-1].$$



Ex:



$\alpha > 0.$

Q: What is $X(s)$?

R.o.c.?

A:

$$X(s) = \int_{-\infty}^{\infty} e^{-\alpha|t|} e^{-st}\, dt = \int_{-\infty}^{0} e^{\alpha t} e^{-st}\, dt + \int_{0}^{\infty} e^{-\alpha t} e^{-st}\, dt.$$

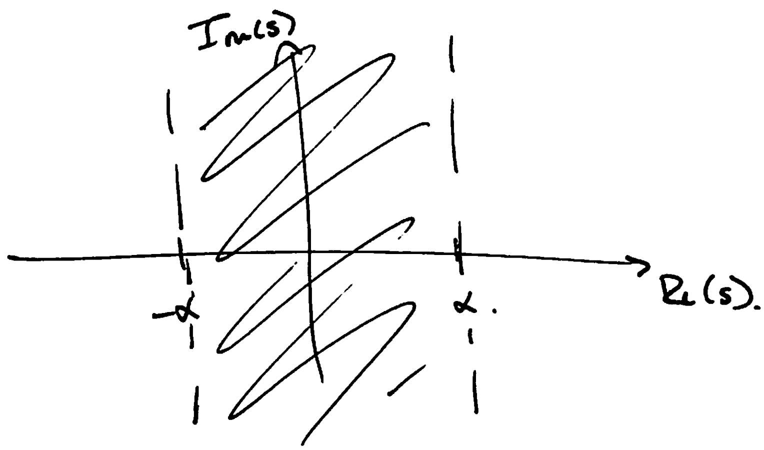
$$= \frac{1}{\alpha - s} e^{(\alpha - s)t}\Big|_{-\infty}^{0} + \frac{-1}{\alpha + s} e^{-(\alpha + s)t}\Big|_{0}^{\infty}$$

$$= \frac{1}{\alpha - s} + \frac{1}{\alpha + s}. \qquad \text{R.o.c } \boxed{-\alpha < Re(s) < \alpha.}$$

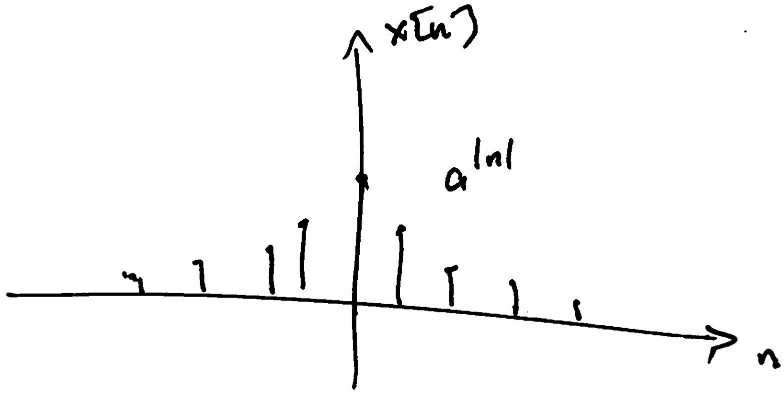
could also get this knowing that $\frac{1}{\alpha + s} \overset{R.S.}{\Longleftrightarrow} e^{-\alpha t} u(t).$

and $\frac{1}{\alpha - s} \overset{L.S.}{\Longleftrightarrow} e^{\alpha t} u(-t).$

R.o.c.:



Ex:



What is $X(z)$?

R.O.C. ?

$$X(z) = \sum_{n=-\infty}^{\infty} a^{|n|} z^{-n} = \sum_{n=-\infty}^{-1} a^{-n} z^{-n} + \sum_{0}^{\infty} a^{n} z^{-n}.$$

$$= \sum_{m=1}^{\infty} a^{m} z^{m} + \sum_{0}^{\infty} \left(\frac{a}{z}\right)^{n} = \frac{az}{1-az} + \frac{1}{1-\frac{a}{z}}.$$

$m=-n.$

R.O.C. $|z| < \frac{1}{a}$. $|z| > a$.