6.003 Recitation 2. 9.9.11.

Russ Tedrake
34-301.

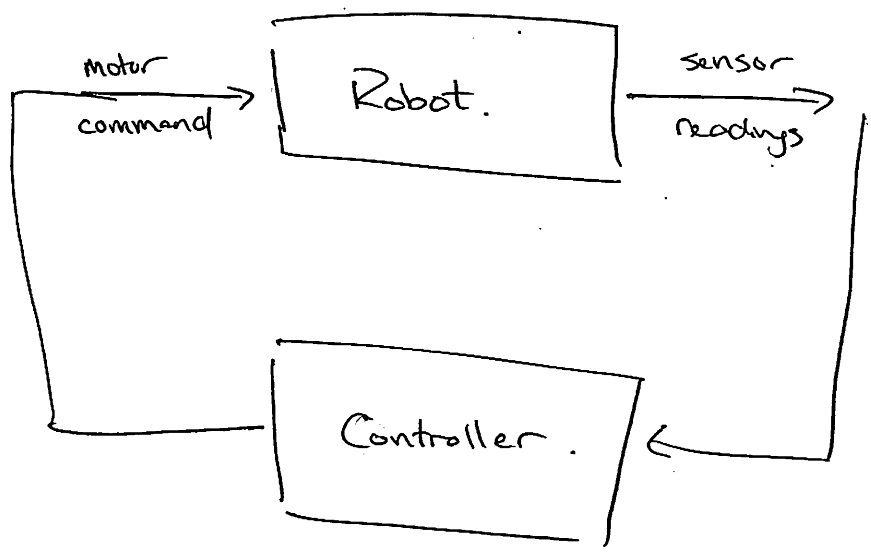
Today: - Quick introductions.
- CT / DT systems.
- Differential equations representation. example
- Difference equation representation. example

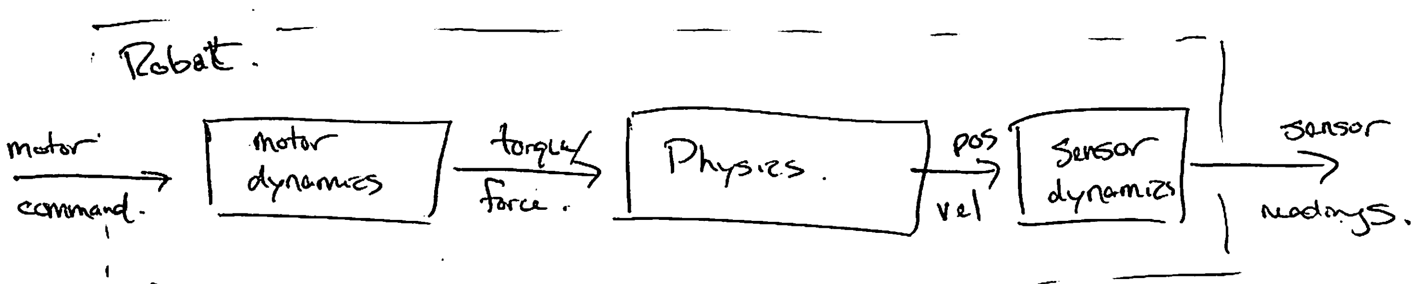
Who am I?

Russ Tedrake russt@mit.edu.

I work on robots.... (robot birds, walking robots, ...)

Designing control systems for these robots is very difficult. I use the "signals + systems" abstraction all of the time.



will do my best to teach you the fundamentals that you need to know to control cool robots.

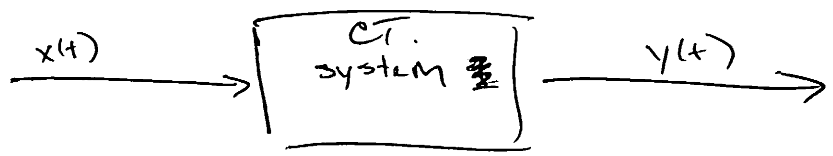
If you don't like robots? (that's like saying you don't like puppies!)

Also useful for signal processing,
communications, ....

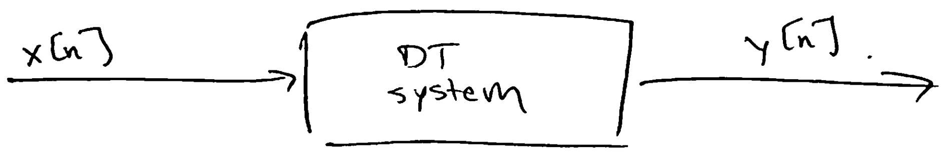
Between Denny, Elfar, and me, you'll hear about all three.

In rec. 1 + lecture 1, we talked mostly about signals.

Today: systems. Please ask lots of questions!




x(t) is a CT signal. (same for y(t)).

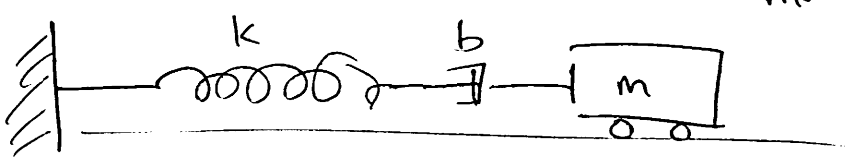



Systems describe what y(t) does as a function of x(t).

How do you describe this?

In practice, it's useful to describe this implicitly.

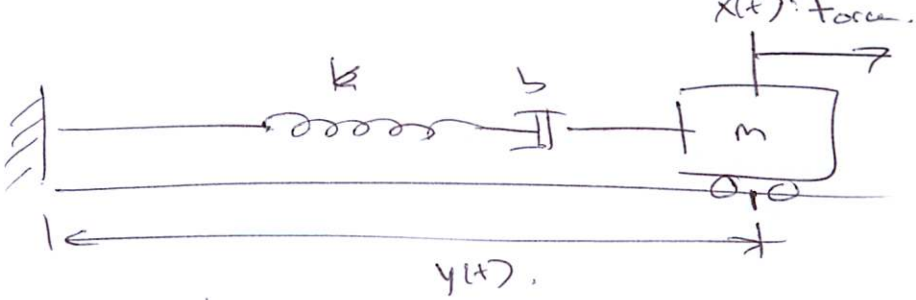
E.G.




mass-spring-damper system.

Q: What is a good input / output description for this system?

Let's use




input. $x(t)$ : horizontal force applied to mass

output: $y(t)$ : position of the mass.

We know how to describe the dynamics implicitly.

$$F = ma \quad \Rightarrow \quad \frac{d^2 y(t)}{dt^2} = f\left(\frac{dy(t)}{dt}, y(t), f(t)\right).$$

It's harder to write ~~y(t) = f(x...)~~

$$Y = f(X).$$

$y(t)$ will depend on $x(t)$ for $t \in [-\infty, t]$

Differential Eq. representation of a system.

we'll practice solving this in a minute (if we have time).

You remember from 18.03.

Also DT systems use difference eqs.

You remember from 601.

I'll try to make connections when possible.

DT system example.

Bank account.

Let's say you have $20K in your bank account.

Let's say 4% interest annually.


Instead of thinking about initial conditions, let's say that the balance is zero until you deposit $20K in year 0.

$x[n]$ money in/out at year n. (units $1K).

$y[n]$ balance at year n.

$$y[n] = 1.04\, y[n-1] + x[n].$$

$x[0] = \cancel{\$20K}\ 10$.

$x[i]\quad i>0 = 0$.

What does $y[n]$ look like?

$y[0] = \cancel{\$20K}$.

$y[1] = 1.04\,(\$20K)$.

$y[2] = (1.04)^2\,(\$20K)$.

$y[n] = (1.04)^n\,(\$20K)$.

f.y.i. $(1.04)^{100} \approx 50$.

compound interest is great.

Realistically, we have expenses. (and almost no income ;-))

Let's say we draw ~~$500~~ $1K / year. from the account.

$x[0] = 20$

$x[1] = -\$1$

$x[i] = -\$1 \quad i > 0.$

Q: Can we live off the interest? or will we eventually hit zero?

A: zero. ~~$500~~ $1K. is 5% of 20k.

$y[0] = 20.$

$y[1] < 20.$

$\Rightarrow$ interest will be less (and withdrawal is the same).

So I'll plummet to zero.

Can we solve for $y[n]$?

Let's guess a form of the solution based on our initial analysis.

$$y[n] = a(1.04)^n + b.$$

$$y[0] = 20 = a + b.$$

$$y[1] = 20 + 0.8 - 1 = 19.8 = a(1.04) + b.$$

will also be correct for $n > 1$

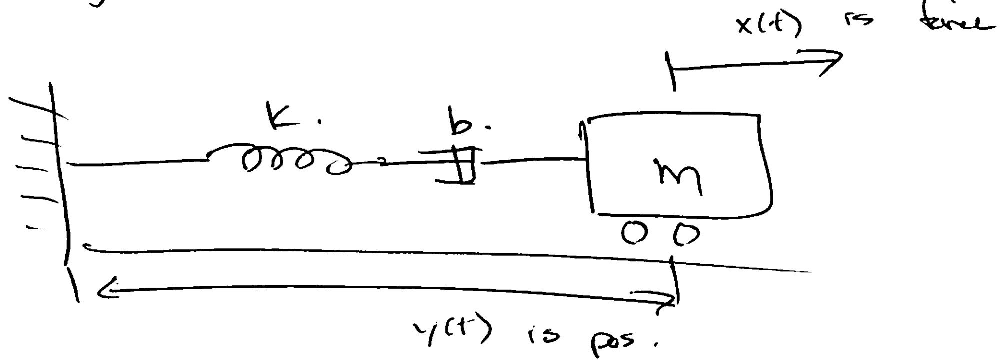
$a = -5$

$b = 25.$

$\Rightarrow y[n] = 25 - 5 \times 1.04^n$

what happens after $y$ crosses $0$?

Ex: Spring - mass - damper.




$$m\ddot{y}(t) + b\dot{y}(t) + ky(t) = x(t).$$

Question: for $x(t) = 0$, what happens?

$b > 0 \Rightarrow$ will eventually stop.

$b^2 > 4mk \Rightarrow$ over-damped . . .

etc.

for under-damped, it will oscillate w/ some frequency til it stops. (natural freq).

Q: $x(t) =$ ~~[illegible]~~ $\cos(\omega t)$.

what happens now?

A: for $b > 0$, will eventually oscillate at $\omega$. (not natural freq).

Let's work that through.

Set $m = 1$ $k = 2$ $b = 1/4$.

First, passive response (homogenous sol.).

$x(t) = 0$.

$$m\ddot{y}(t) + b\dot{y}(t) + k\,y(t) = 0.$$

Assume
$$y(t) = e^{st}.$$
$$\dot{y}(t) = s e^{st}$$
$$\ddot{y}(t) = s^2 e^{st}.$$

$$(ms^2 + bs + k)e^{st} = 0.$$

$$e^{st} \neq 0 \Rightarrow (ms^2 + bs + k) = 0.$$

solve quadratic eq.

$$s_{1,2} = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}.$$ ← complex if $b^2 < 4mk$.

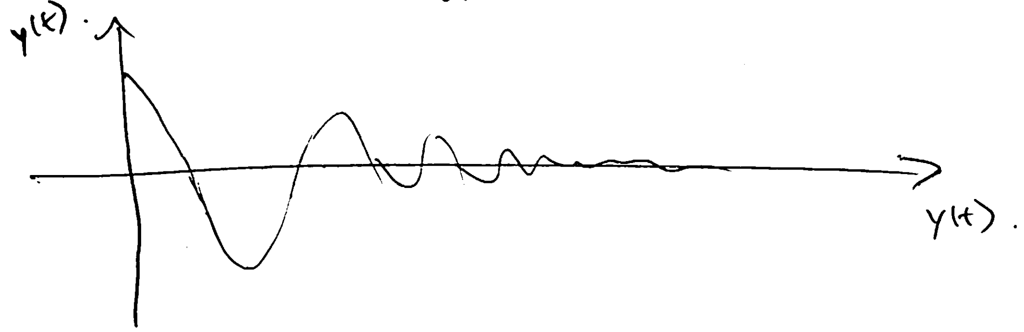
$\cancel{(s-s_1)(s-s_2)=0}$

$$s_{1,2} = \frac{-\frac{1}{4} \pm \sqrt{\frac{1}{16} - 8}}{2} = -\frac{1}{8} \pm \sqrt{\frac{1}{32} - 4}.$$

$$\Rightarrow y(t) = a e^{s_1 t} + b e^{s_2 t}.$$

if $y(0) = 2$, $\dot{y}(0) = 0$.

solve for $a, b$.




Q: $s_1$ complex. does it make sense to have $y(t)$, the position, as a complex numbers?

A: No: complex conj pairs cancel out. $y(t)$ is real.

w/ forcing.

~~recall~~ $x(t) = \cos(\omega t)$.

recall $\cos(\omega t) = \left(e^{j\omega t} + e^{-j\omega t}\right)/2$.

Solution will be of the form.

$$y(t) = \underbrace{a e^{s_1 t} + b e^{s_2 t}}_{} + \underbrace{c e^{j\omega t} + d e^{-j\omega t}}_{}.$$

if $b>0$. these terms will die out.

these persist.

In fact, if zero initial conditions,

then ~~Pass~~ LTI systems output ~~will~~ must match frequency of input. (modified by freq response).