6.003: Signals and Systems

Applications of Fourier Transforms

November 17, 2011

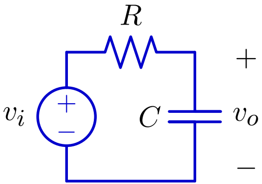
Filtering

Notion of a filter.

LTI systems

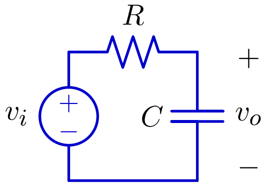
- cannot create new frequencies.
- can only scale magnitudes and shift phases of existing components.

Example: Low-Pass Filtering with an RC circuit

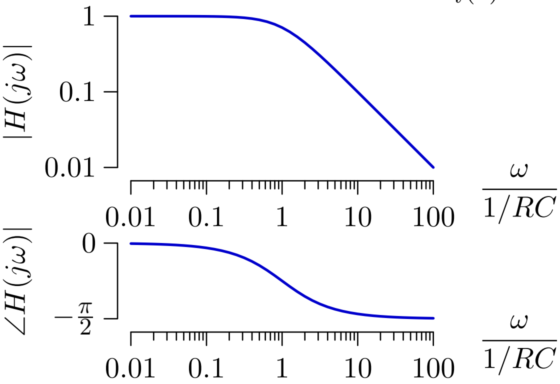


Lowpass Filter

Calculate the frequency response of an RC circuit.

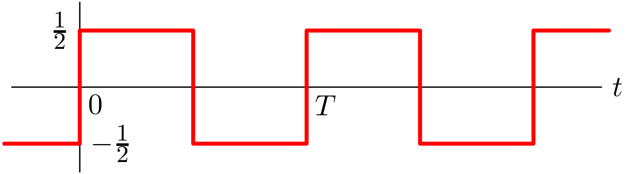


$$
\begin{aligned}
\text{KVL:} \quad v_i(t) &= Ri(t) + v_o(t) \\
\text{C:} \quad i(t) &= C\dot{v}_o(t) \\
\text{Solving:} \quad v_i(t) &= RC\dot{v}_o(t) + v_o(t) \\
V_i(s) &= (1 + sRC)V_o(s) \\
H(s) &= \frac{V_o(s)}{V_i(s)} = \frac{1}{1 + sRC}
\end{aligned}
$$

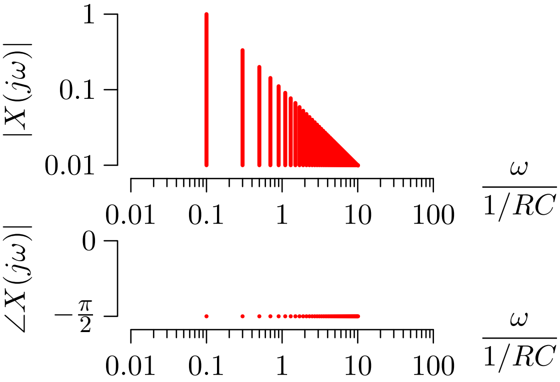


Lowpass Filtering

Let the input be a square wave.

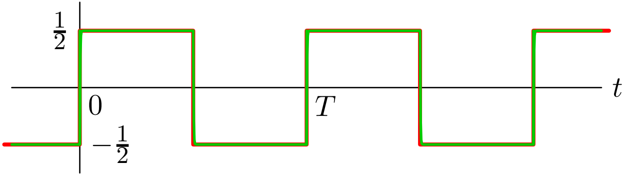


$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 k t} \quad ; \quad \omega_0 = \frac{2\pi}{T}$$

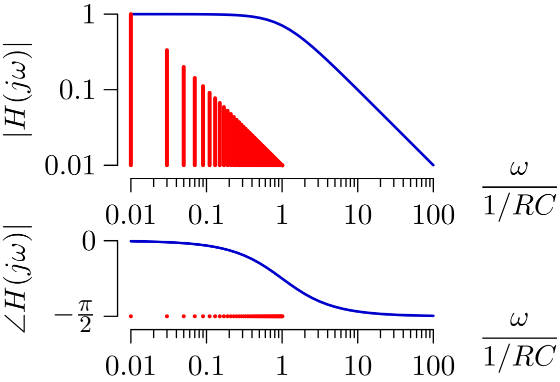


Lowpass Filtering

Low frequency square wave: $\omega_0 << 1/RC$.

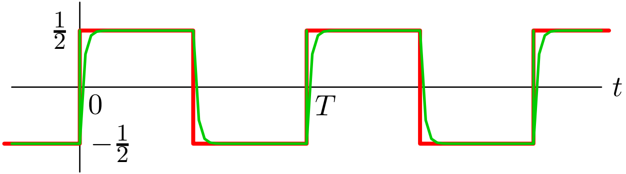


$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 kt} \quad ; \qquad \omega_0 = \frac{2\pi}{T}$$

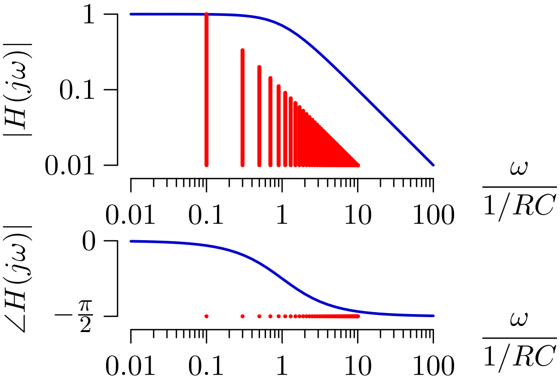


Lowpass Filtering

Higher frequency square wave: $\omega_0 < 1/RC$.

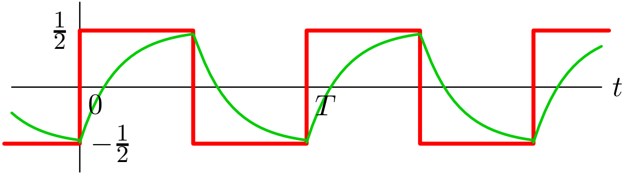


$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 kt} \quad ; \qquad \omega_0 = \frac{2\pi}{T}$$

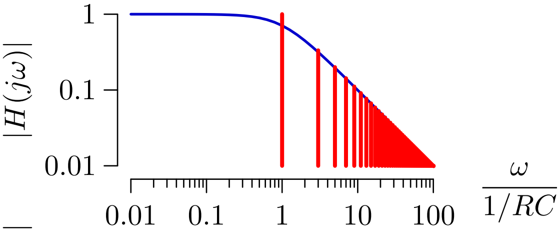


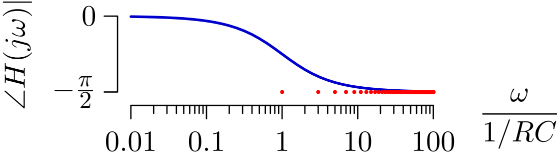
Lowpass Filtering

Still higher frequency square wave: $\omega_0 = 1/RC$.



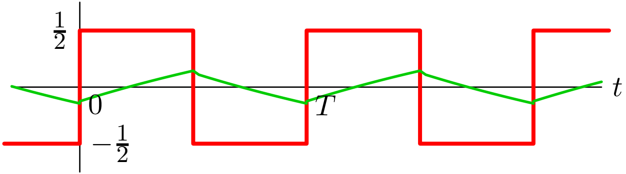
$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 kt} \quad ; \qquad \omega_0 = \frac{2\pi}{T}$$



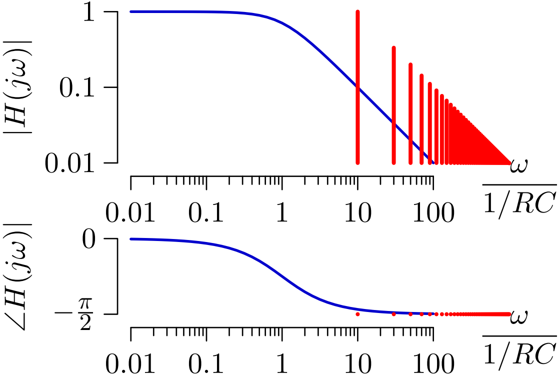


Lowpass Filtering

High frequency square wave: $\omega_0 > 1/RC$.

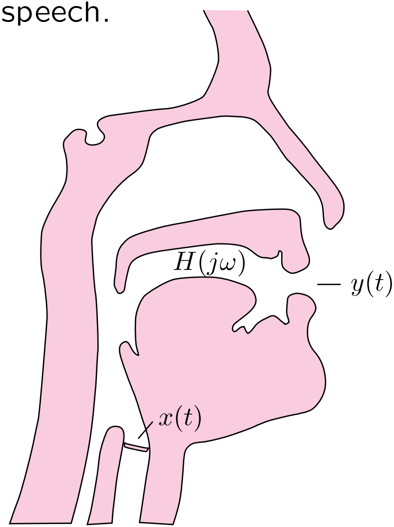


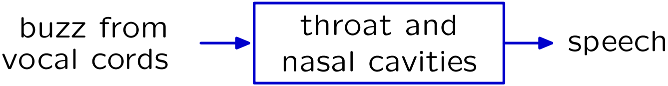
$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 kt} \quad ; \qquad \omega_0 = \frac{2\pi}{T}$$



Source-Filter Model of Speech Production

Vibrations of the vocal cords are "filtered" by the mouth and nasal cavities to generate speech.





Filtering

LTI systems "filter" signals based on their frequency content.

Fourier transforms represent signals as sums of complex exponentials.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

Complex exponentials are eigenfunctions of LTI systems.

$$e^{j\omega t} \rightarrow H(j\omega)e^{j\omega t}$$

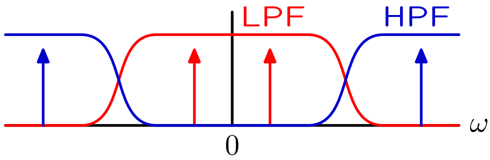
LTI systems "filter" signals by adjusting the amplitudes and phases of each frequency component.

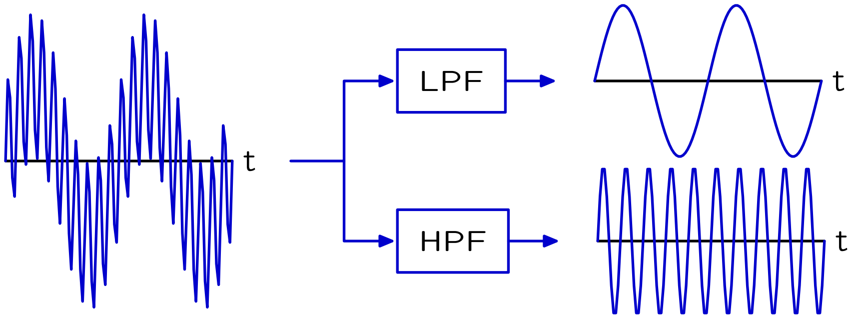
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \;\rightarrow\; y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega)X(j\omega)e^{j\omega t}d\omega$$

Filtering

Systems can be designed to selectively pass certain frequency bands.

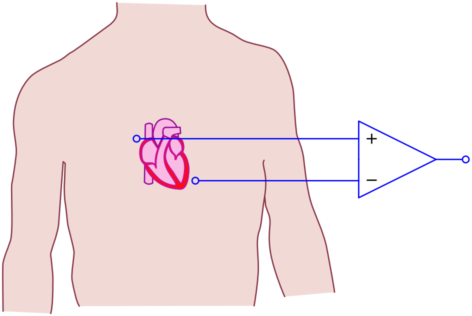
Examples: low-pass filter (LPF) and high-pass filter (HPF).

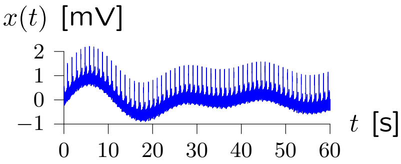




Filtering Example: Electrocardiogram

An electrocardiogram is a record of electrical potentials that are generated by the heart and measured on the surface of the chest.





ECG and analysis by T. F. Weiss

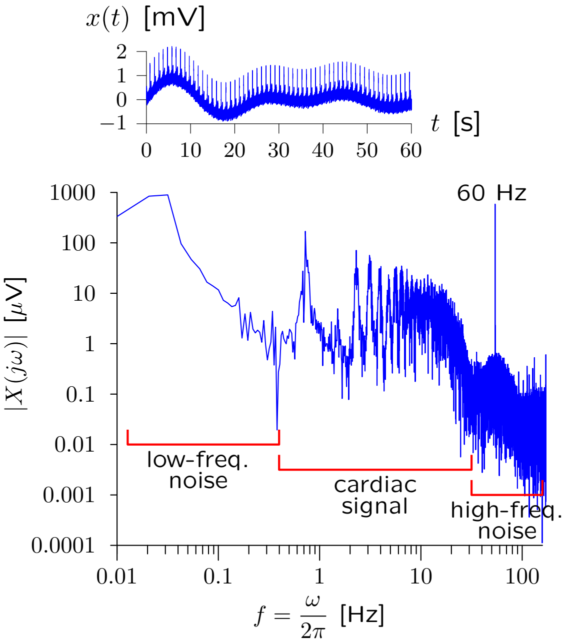
Filtering Example: Electrocardiogram

In addition to electrical responses of heart, electrodes on the skin also pick up other electrical signals that we regard as "noise."

We wish to design a filter to eliminate the noise.

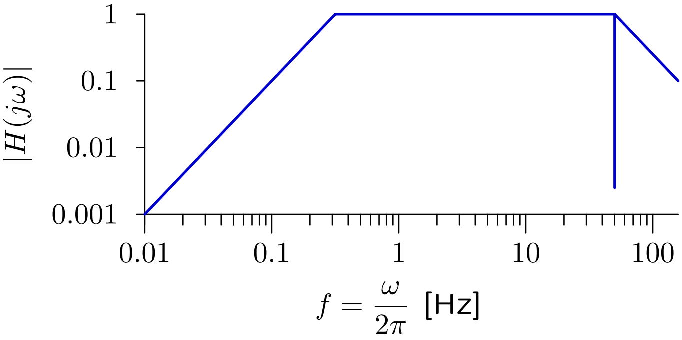
Filtering Example: Electrocardiogram

We can identify "noise" using the Fourier transform.



Filtering Example: Electrocardiogram

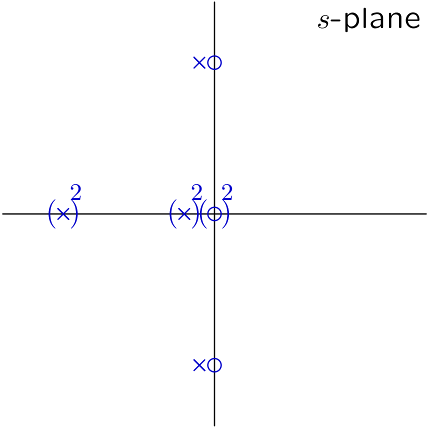
Filter design: low-pass flter + high-pass filter + notch.



Electrocardiogram: Check Yourself

Which poles and zeros are associated with

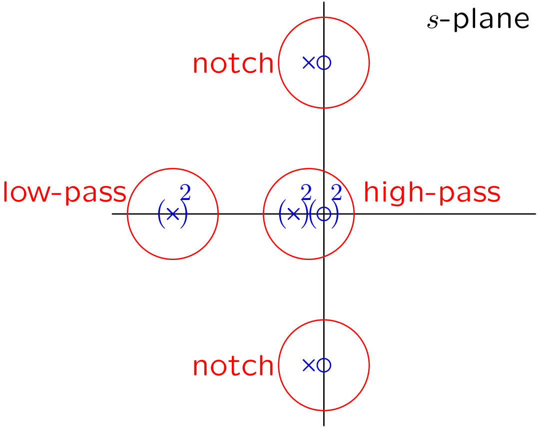
- the high-pass filter?
- the low-pass filter?
- the notch filter?



Electrocardiogram: Check Yourself

Which poles and zeros are associated with

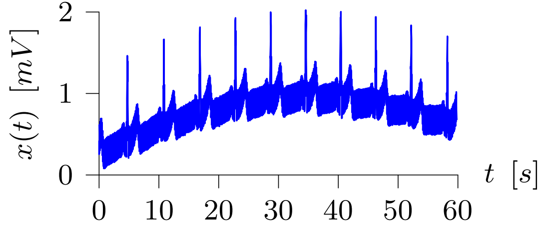
- the high-pass filter?
- the low-pass filter?
- the notch filter?



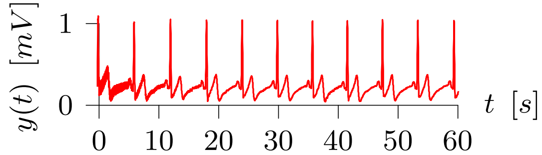
Filtering Example: Electrocardiogram

Filtering is a simple way to reduce unwanted noise.

Unfiltered ECG

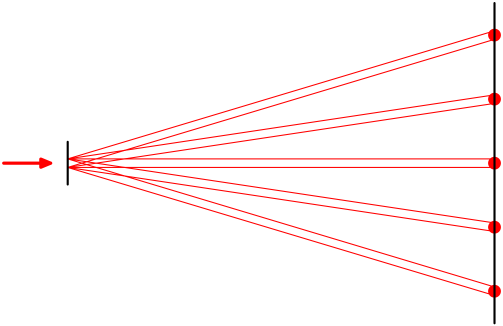


Filtered ECG



Fourier Transforms in Physics: Diffraction

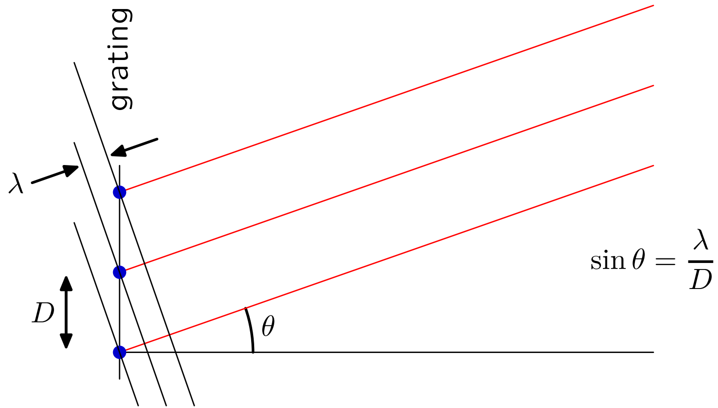
A diffraction grating breaks a laser beam input into multiple beams.



Demonstration.

Fourier Transforms in Physics: Diffraction

Multiple beams result from periodic structure of grating (period D).



Viewed at a distance from angle θ, scatterers are separated by $D\sin\theta$.

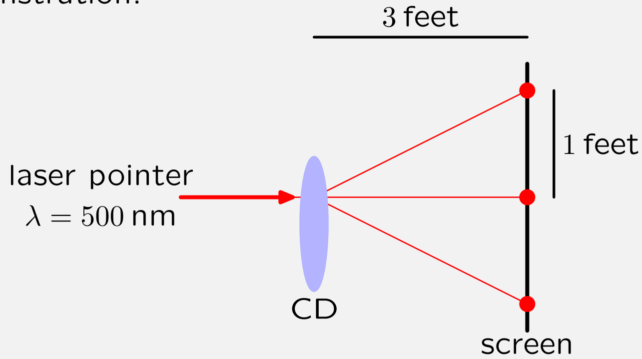
Constructive interference if $D\sin\theta = n\lambda$, i.e., if $\sin\theta = \frac{n\lambda}{D}$
$\rightarrow$ periodic array of dots in the far field

Fourier Transforms in Physics: Diffraction

CD demonstration.

Check Yourself

CD demonstration.



What is the spacing of the tracks on the CD?

1. 160 nm
2. 1600 nm
3. 16μm
4. 160μm

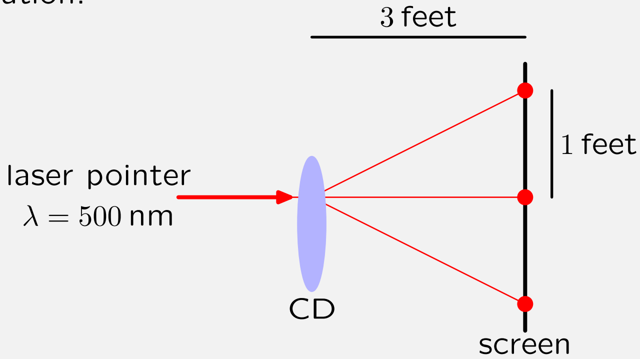
Check Yourself

What is the spacing of the tracks on the CD?

grating	$\tan\theta$	θ	$\sin\theta$	$D = \dfrac{500\,\text{nm}}{\sin\theta}$	manufacturing spec.
CD	$\frac{1}{3}$	0.32	0.31	$1613\,\text{nm}$	$1600\,\text{nm}$

Check Yourself

Demonstration.



What is the spacing of the tracks on the CD? 2.

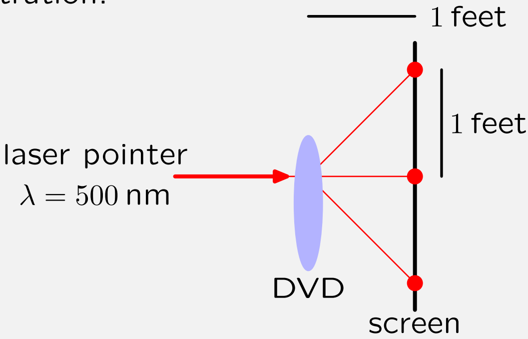
1. 160 nm   2. 1600 nm   3. 16μm   4. 160μm

Fourier Transforms in Physics: Diffraction

DVD demonstration.

Check Yourself

DVD demonstration.



What is track spacing on DVD divided by that for CD?

1. $4\times$
2. $2\times$
3. $\frac{1}{2}\times$
4. $\frac{1}{4}\times$

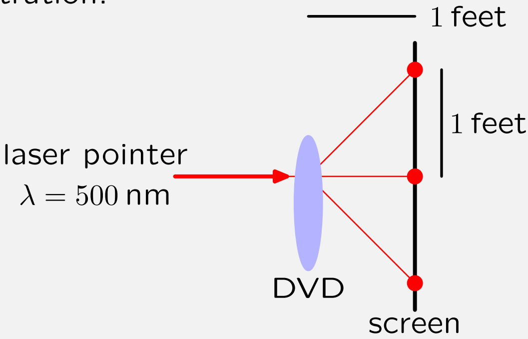
Check Yourself

What is spacing of tracks on DVD divided by that for CD?

grating	$\tan\theta$	θ	$\sin\theta$	$D = \dfrac{500\,\text{nm}}{\sin\theta}$	manufacturing spec.
CD	$\frac{1}{3}$	0.32	0.31	$1613\,\text{nm}$	$1600\,\text{nm}$
DVD	1	0.78	0.71	$704\,\text{nm}$	$740\,\text{nm}$

Check Yourself

DVD demonstration.

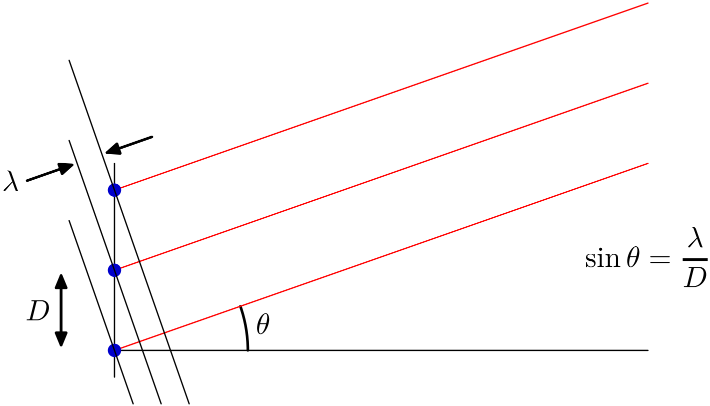


What is track spacing on DVD divided by that for CD? 3

1. $4\times$
2. $2\times$
3. $\frac{1}{2}\times$
4. $\frac{1}{4}\times$

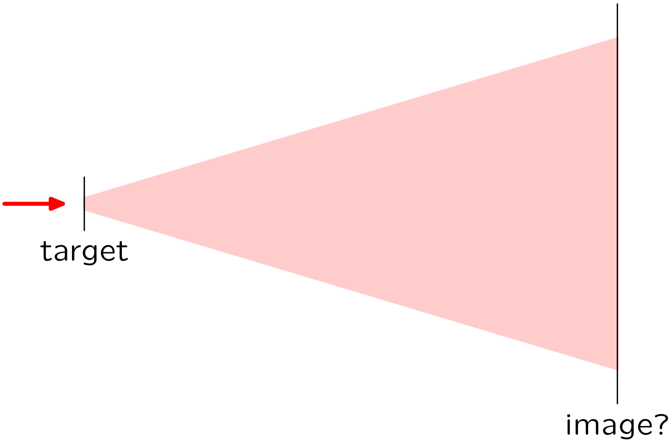
Fourier Transforms in Physics: Diffraction

Macroscopic information in the far field provides microscopic (invisible) information about the grating.



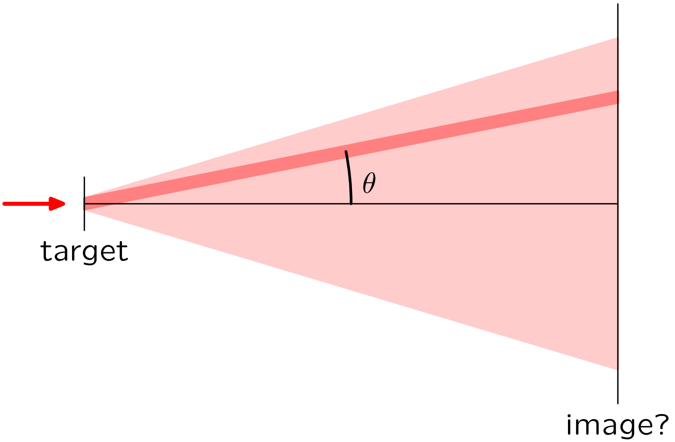
Fourier Transforms in Physics: Crystallography

What if the target is more complicated than a grating?



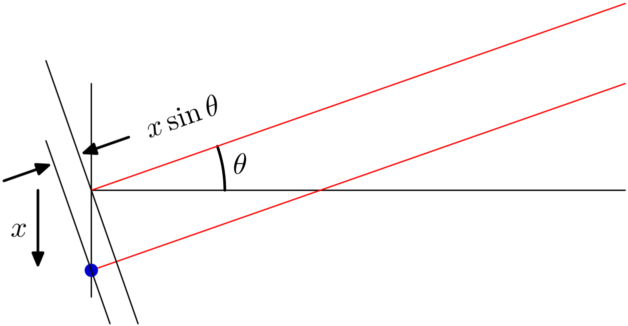
Fourier Transforms in Physics: Crystallography

Part of image at angle θ has contributions for all parts of the target.



Fourier Transforms in Physics: Crystallography

The phase of light scattered from different parts of the target undergo different amounts of phase delay.



Phase at a point x is delayed (i.e., negative) relative to that at 0:

$$\phi = -2\pi\frac{x\sin\theta}{\lambda}$$

Fourier Transforms in Physics: Crystallography

Total light $F(\theta)$ at angle θ is integral of light scattered from each part of target $f(x)$, appropriately shifted in phase.

$$F(\theta) = \int f(x)\, e^{-j2\pi \frac{x \sin\theta}{\lambda}} dx$$

Assume small angles so $\sin\theta \approx \theta$.

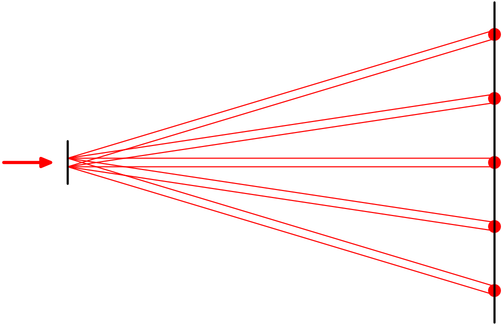
Let $\omega = 2\pi\frac{\theta}{\lambda}$, then the pattern of light at the detector is

$$F(\omega) = \int f(x)\, e^{-j\omega x} dx$$

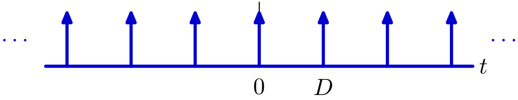
which is the Fourier transform of $f(x)$!

Fourier Transforms in Physics: Diffraction

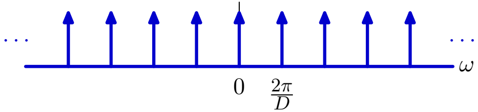
Fourier transform relation between structure of object and far-field intensity pattern.



grating $\approx$ impulse train with pitch D



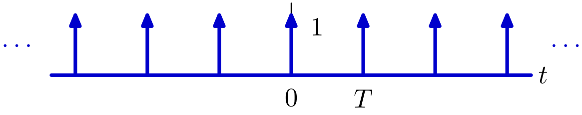
far-field intensity $\approx$ impulse train with reciprocal pitch $\propto \frac{\lambda}{D}$



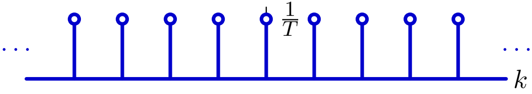
Impulse Train

The Fourier transform of an impulse train is an impulse train.

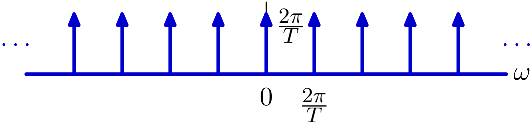
$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$



$$a_k = \frac{1}{T} \quad \forall \ k$$



$$X(j\omega) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{T} \delta(\omega - k\frac{2\pi}{T})$$

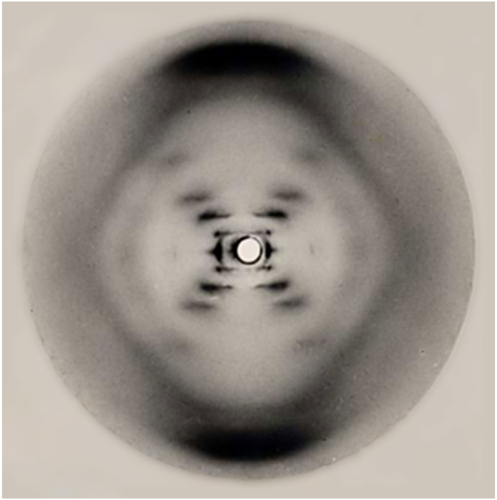


Two Dimensions

Demonstration: 2D grating.

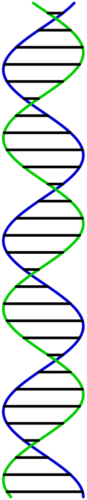
An Historic Fourier Transform

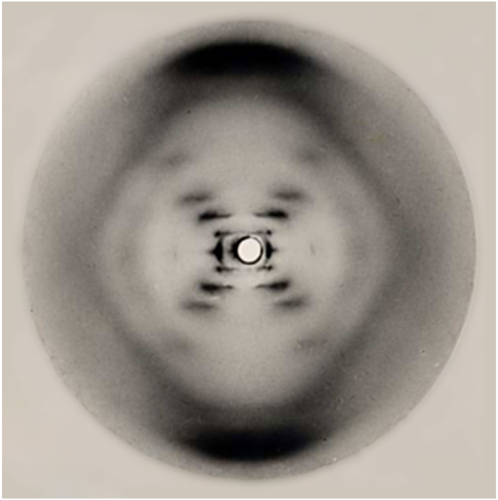
Taken by Rosalind Franklin, this image sparked Watson and Crick's insight into the double helix.



An Historic Fourier Transform

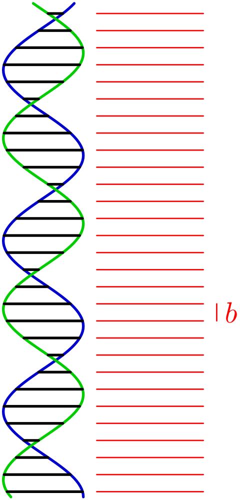
This is an x-ray crystallographic image of DNA, and it shows the Fourier transform of the structure of DNA.

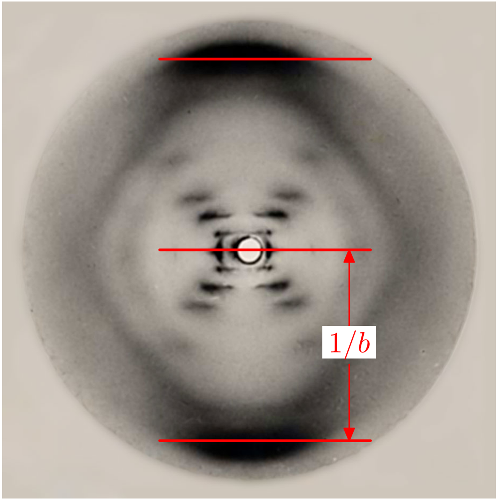




An Historic Fourier Transform

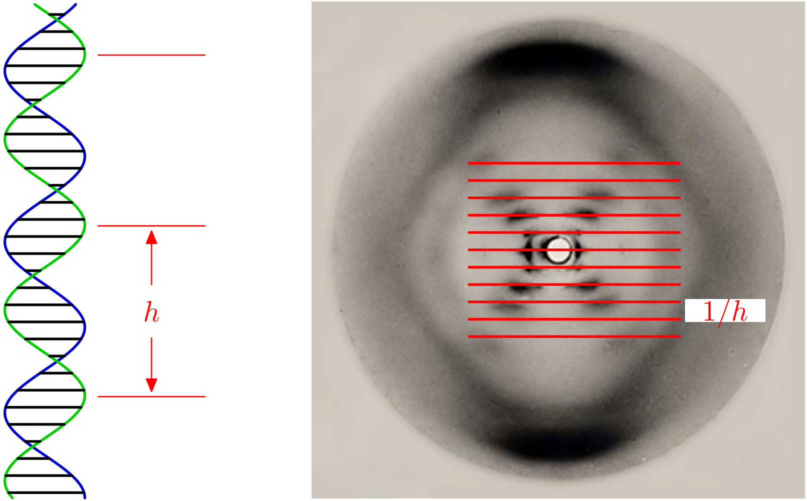
High-frequency bands indicate repeating structure of base pairs.





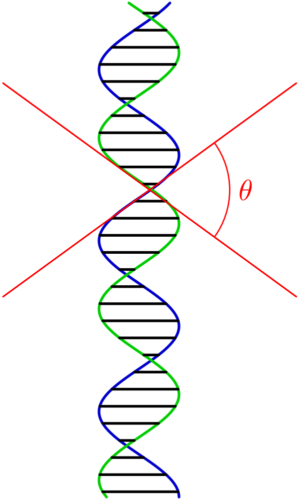
An Historic Fourier Transform

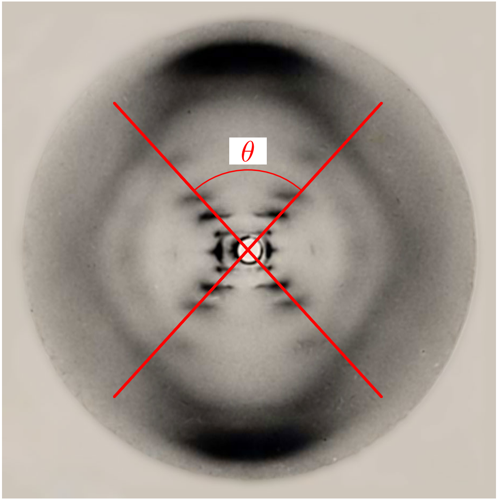
Low-frequency bands indicate a lower frequency repeating structure.



An Historic Fourier Transform

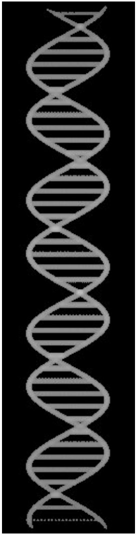
Tilt of low-frequency bands indicates tilt of low-frequency repeating structure: the double helix!

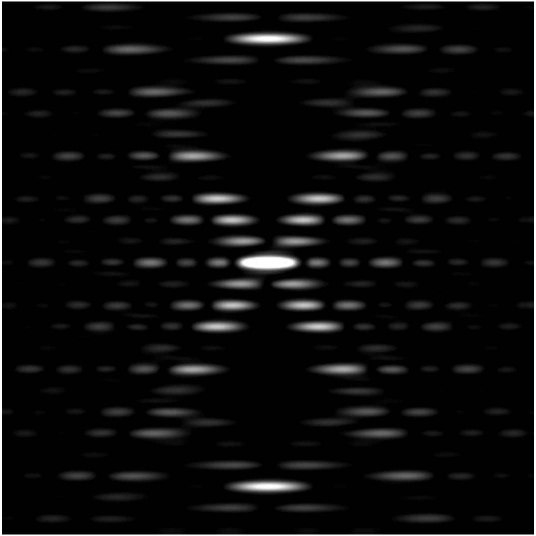




Simulation

Easy to calculate relation between structure and Fourier transform.





Fourier Transform Summary

Represent signals by their frequency content.

Key to “filtering,” and to signal-processing in general.

Important in many physical phenomenon: x-ray crystallography.