6.003: Signals and Systems

Relations among Fourier Representations

November 15, 2011

Mid-term Examination #3

Wednesday, November 16, 7:30-9:30pm, Walker (50-340)

No recitations on the day of the exam.

Coverage: Lectures 1–18
Recitations 1–16
Homeworks 1–10

Homework 10 will not be collected or graded.
Solutions are posted.

Closed book: 3 pages of notes ($8\frac{1}{2} \times 11$ inches; front and back).

No calculators, computers, cell phones, music players, or other aids.

Designed as 1-hour exam; two hours to complete.

Prior term midterm exams have been posted on the 6.003 website.

Fourier Representations

We've seen a variety of Fourier representations:

- CT Fourier series
- CT Fourier transform
- DT Fourier series
- DT Fourier transform

Today: relations among the four Fourier representations.

Four Fourier Representations

We have discussed four closely related Fourier representations.

DT Fourier Series

$$a_k = a_{k+N} = \frac{1}{N} \sum_{n=<N>} x[n] e^{-j\frac{2\pi}{N}kn}$$

$$x[n] = x[n+N] = \sum_{k=<N>} a_k e^{j\frac{2\pi}{N}kn}$$

DT Fourier transform

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

$$x[n] = \frac{1}{2\pi} \int_{<2\pi>} X(e^{j\Omega}) e^{j\Omega n} d\Omega$$

CT Fourier Series

$$a_k = \frac{1}{T} \int_T x(t) e^{-j\frac{2\pi}{T}kt} dt$$

$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt}$$

CT Fourier transform

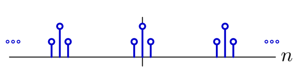
$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

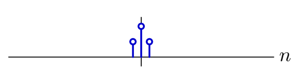
Four Types of "Time"

discrete vs. continuous (↕) and periodic vs aperiodic (↔)

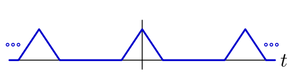
DT Fourier Series



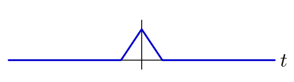
DT Fourier transform



CT Fourier Series



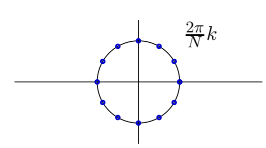
CT Fourier transform



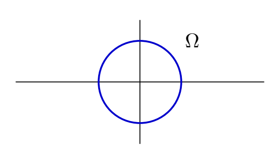
Four Types of "Frequency"

discrete vs. continuous (↔) and periodic vs aperiodic (↕)

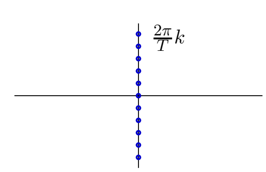
DT Fourier Series



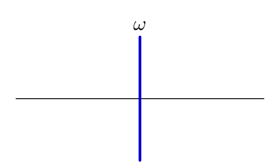
DT Fourier transform



CT Fourier Series



CT Fourier transform



Relation between Fourier Series and Transform

A periodic signal can be represented by a Fourier series or by an equivalent Fourier transform.

Series: represent periodic signal as weighted sum of harmonics

$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt} \ ; \qquad \omega_0 = \frac{2\pi}{T}$$

Relation between Fourier Series and Transform

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$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt} \ ; \qquad \omega_0 = \frac{2\pi}{T}$$

The Fourier transform of a sum is the sum of the Fourier transforms:

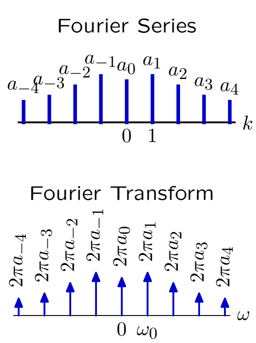
$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

Therefore periodic signals can be equivalently represented as Fourier transforms (with impulses!).

Relation between Fourier Series and Transform

A periodic signal can be represented by a Fourier series or by an equivalent Fourier transform.

$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt} \quad \leftrightarrow$$

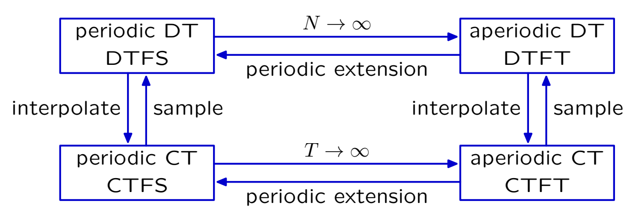


Relations among Fourier Representations

Explore other relations among Fourier representations.

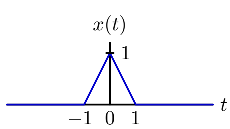
Start with an aperiodic CT signal. Determine its Fourier transform.

Convert the signal so that it can be represented by alternate Fourier representations and compare.



Start with the CT Fourier Transform

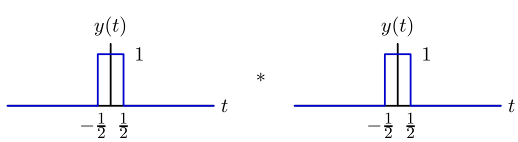
Determine the Fourier transform of the following signal.



Could calculate Fourier transform from the definition.

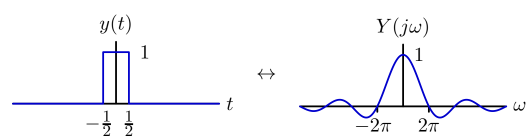
$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{j\omega t} dt$$

Easier to calculate $x(t)$ by convolution of two square pulses:

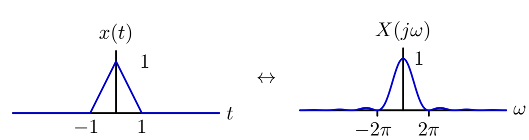


Start with the CT Fourier Transform

The transform of $y(t)$ is $Y(j\omega) = \frac{\sin(\omega/2)}{\omega/2}$



so the transform of $x(t) = (y * y)(t)$ is $X(j\omega) = Y(j\omega) \times Y(j\omega)$.

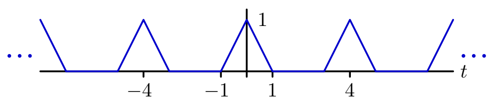


Relation between Fourier Transform and Series

What is the effect of making a signal periodic in time?

Find Fourier transform of periodic extension of $x(t)$ to period $T = 4$.

$$z(t) = \sum_{k=-\infty}^{\infty} x(t+4k)$$

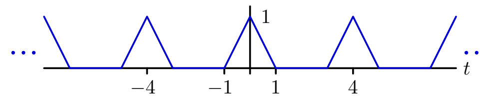


Could calculate $Z(j\omega)$ for the definition … ugly.

Relation between Fourier Transform and Series

Easier to calculate $z(t)$ by convolving $x(t)$ with an impulse train.

$$z(t) = \sum_{k=-\infty}^{\infty} x(t+4k)$$



$$z(t) = \sum_{k=-\infty}^{\infty} x(t+4k) = (x * p)(t)$$

where

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t+4k)$$

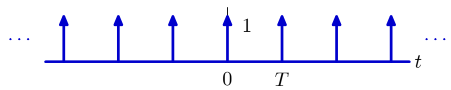
Then

$$Z(j\omega) = X(j\omega) \times P(j\omega)$$

Check Yourself

What's the Fourier transform of an impulse train?

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT)$$



Relation between Fourier Transform and Series

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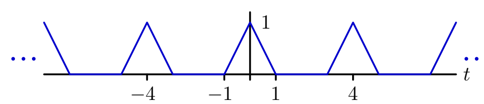


$$z(t) = \sum_{k=-\infty}^{\infty} x(t+4k) = (x * p)(t)$$

where

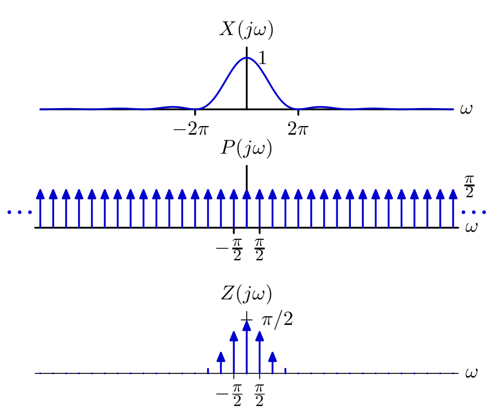
$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t+4k)$$

Then

$$Z(j\omega) = X(j\omega) \times P(j\omega)$$

Relation between Fourier Transform and Series

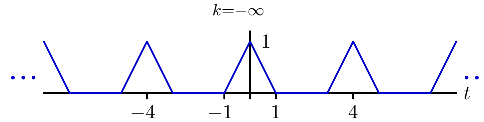
Convolving in time corresponds to multiplying in frequency.

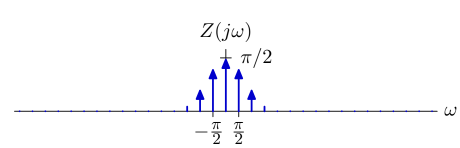


Relation between Fourier Transform and Series

The Fourier transform of a periodically extended function is a discrete function of frequency ω.

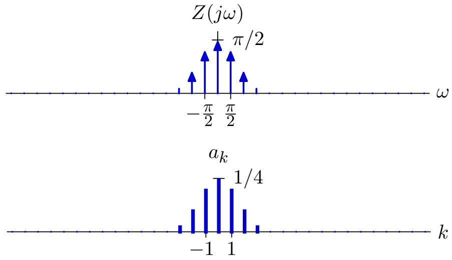
$$z(t) = \sum_{k=-\infty}^{\infty} x(t+4k)$$





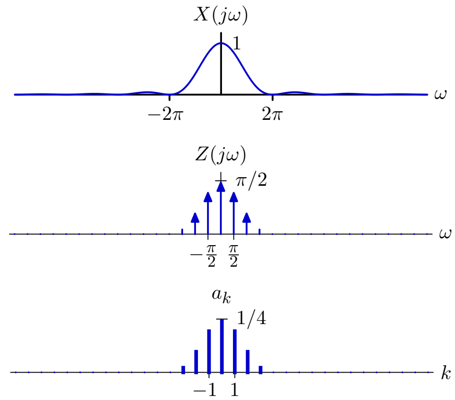
Relation between Fourier Transform and Series

The weight (area) of each impulse in the Fourier transform of a periodically extended function is 2π times the corresponding Fourier series coefficient.



Relation between Fourier Transform and Series

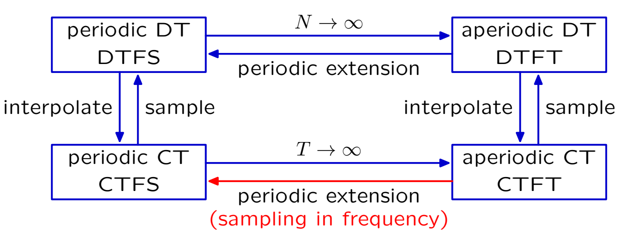
The effect of periodic extension of $x(t)$ to $z(t)$ is to sample the frequency representation.



Relation between Fourier Transform and Series

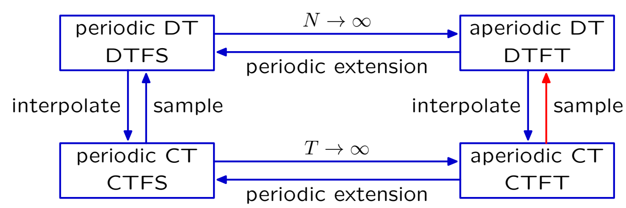
Periodic extension of CT signal $\rightarrow$ discrete function of frequency.

Periodic extension
 = convolving with impulse train in time
  = multiplying by impulse train in frequency
   $\rightarrow$ sampling in frequency



Relations among Fourier Representations

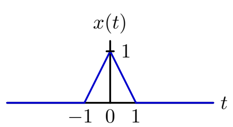
Compare to sampling in time.



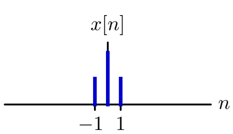
Relations between CT and DT transforms

Sampling a CT signal generates a DT signal.

$$x[n] = x(nT)$$



Take $T = \frac{1}{2}$.

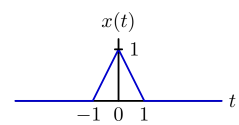


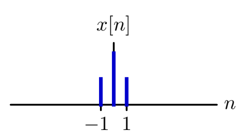
What is the effect on the frequency representation?

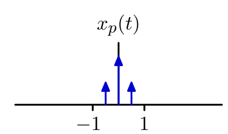
Relations between CT and DT transforms

We can generate a signal with the same shape by multiplying $x(t)$ by an impulse train with $T = \frac{1}{2}$.

$$x_p(t) = x(t) \times p(t) \quad \text{where} \quad p(t) = \sum_{k=-\infty}^{\infty} \delta(t + kT)$$



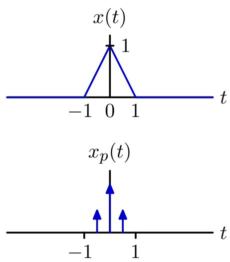




Relations between CT and DT transforms

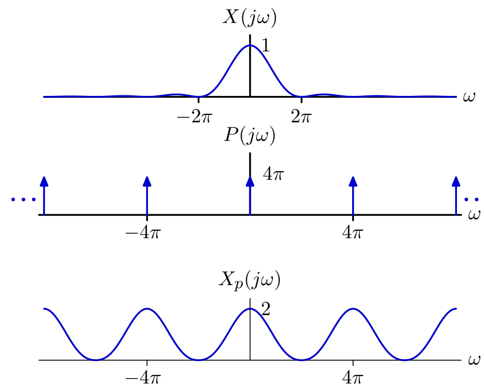
We can generate a signal with the same shape by multiplying $x(t)$ by an impulse train with $T = \frac{1}{2}$.

$$x_p(t) = x(t) \times p(t) \quad \text{where} \quad p(t) = \sum_{k=-\infty}^{\infty} \delta(t + kT)$$



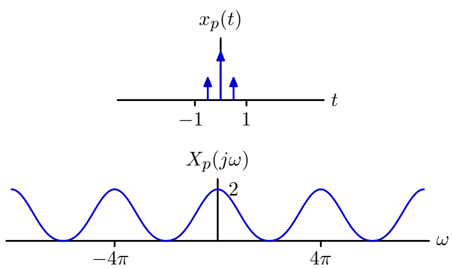
Relations between CT and DT transforms

Multiplying $x(t)$ by an impulse train in time is equivalent to convolving $X(j\omega)$ by an impulse train in frequency (then $\div 2\pi$).



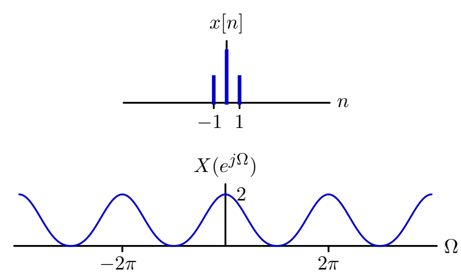
Relations between CT and DT transforms

Fourier transform of sampled signal $x_p(t)$ is periodic in ω, period 4π.



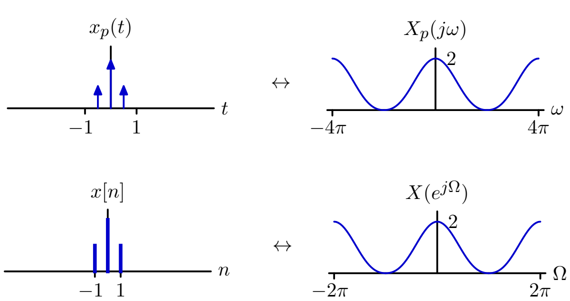
Relations between CT and DT transforms

Fourier transform of sampled signal $x_p(t)$ has same shape as DT Fourier transform of $x[n]$.



DT Fourier transform

CT Fourier transform of sampled signal $x_p(t)$ = DT Fourier transform of samples $x[n]$ where $\Omega = \omega T$, i.e., $X(j\omega) = X(e^{j\Omega})\big|_{\Omega=\omega T}$.



$\Omega = \omega T = \frac{1}{2}\omega$

Relation between CT and DT Fourier Transforms

Compare the definitions:

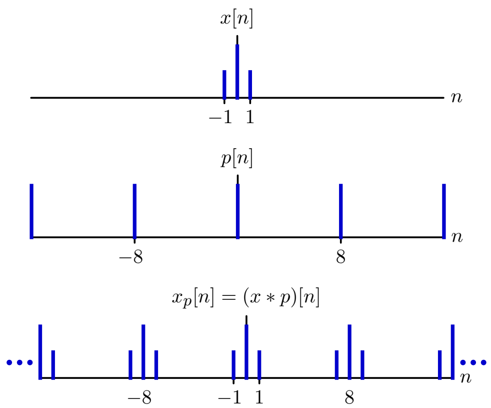
$$X(e^{j\Omega}) = \sum_n x[n] e^{-j\Omega n}$$

$$\begin{aligned} X_p(j\omega) &= \int x_p(t) e^{-j\omega t} dt \\ &= \int \sum_n x[n] \delta(t - nT) e^{-j\omega t} dt \\ &= \sum_n x[n] \int \delta(t - nT) e^{-j\omega t} dt \\ &= \sum_n x[n] e^{-j\omega n T} \end{aligned}$$

$\Omega = \omega T$

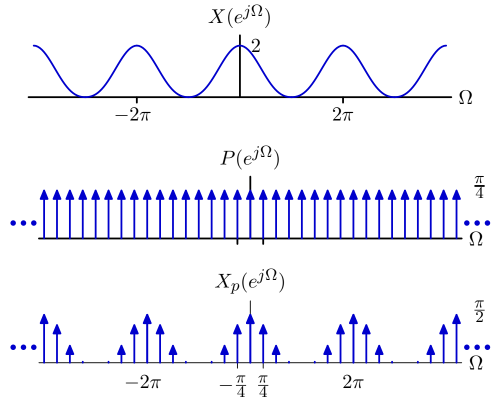
Relation Between DT Fourier Transform and Series

Periodic extension of a DT signal is equivalent to convolution of the signal with an impulse train.



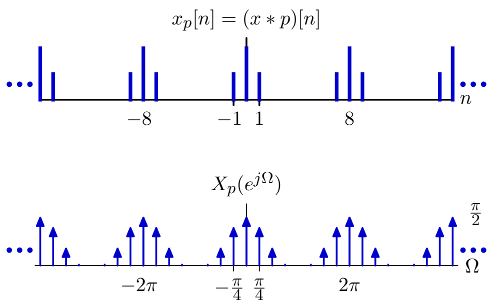
Relation Between DT Fourier Transform and Series

Convolution by an impulse train in time is equivalent to multiplication by an impulse train in frequency.



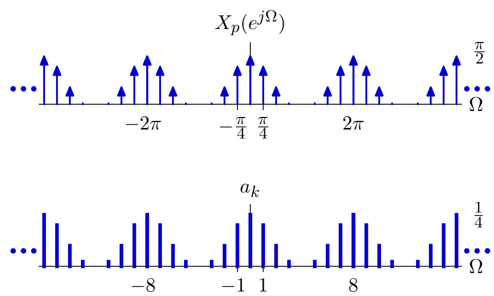
Relation Between DT Fourier Transform and Series

Periodic extension of a discrete signal ($x[n]$) results in a signal ($x_p[n]$) that is both periodic and discrete. Its transform ($X_p(e^{j\Omega})$) is also periodic and discrete.



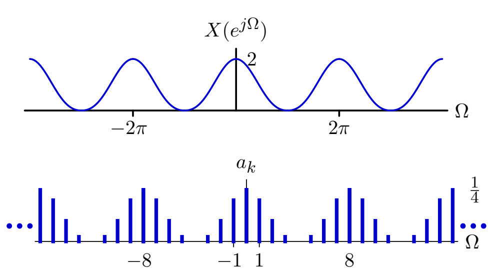
Relation Between DT Fourier Transform and Series

The weight of each impulse in the Fourier transform of a periodically extended function is 2π times the corresponding Fourier series coefficient.



Relation between Fourier Transforms and Series

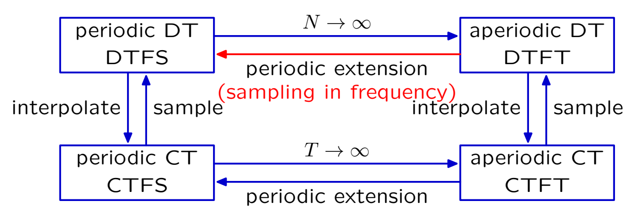
The effect of periodic extension was to sample the frequency representation.



Relation between Fourier Transforms and Series

Periodic extension of a DT signal produces a discrete function of frequency.

Periodic extension
  = convolving with impulse train in time
    = multiplying by impulse train in frequency
      → sampling in frequency



Four Fourier Representations

Underlying structure → view as one transform, not four.

DT Fourier Series

$$a_k = a_{k+N} = \frac{1}{N}\sum_{n=<N>} x[n]e^{-j\frac{2\pi}{N}kn}$$

$$x[n] = x[n+N] = \sum_{k=<N>} a_k e^{j\frac{2\pi}{N}kn}$$

DT Fourier transform

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

$$x[n] = \frac{1}{2\pi}\int_{<2\pi>} X(e^{j\Omega})e^{j\Omega n}d\Omega$$

CT Fourier Series

$$a_k = \frac{1}{T}\int_T x(t)e^{-j\frac{2\pi}{T}kt}dt$$

$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt}$$

CT Fourier transform

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$