6.003: Signals and Systems

Fourier Representations

October 27, 2011

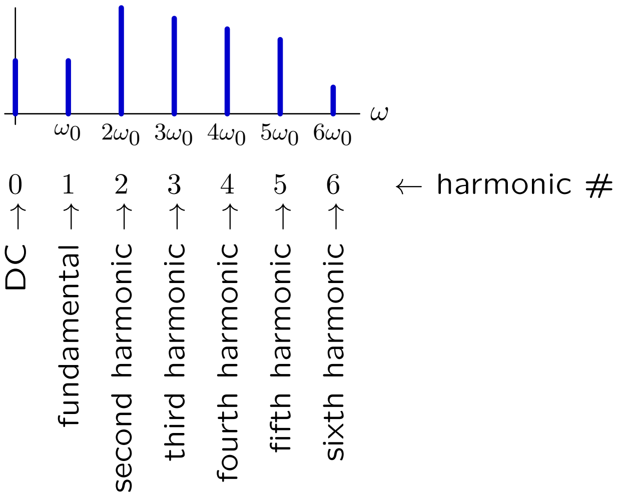
Fourier Representations

Fourier series represent **signals** in terms of **sinusoids**.

$\rightarrow$ leads to a new representation for **systems** as **filters**.

Fourier Series

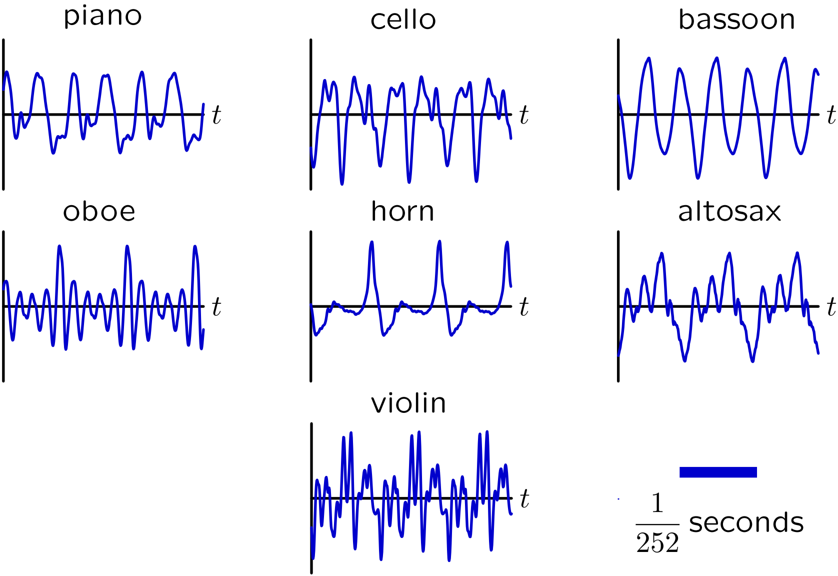
Representing signals by their harmonic components.



Musical Instruments

Harmonic content is natural way to describe some kinds of signals.

Ex: musical instruments (http://theremin.music.uiowa.edu/MIS)



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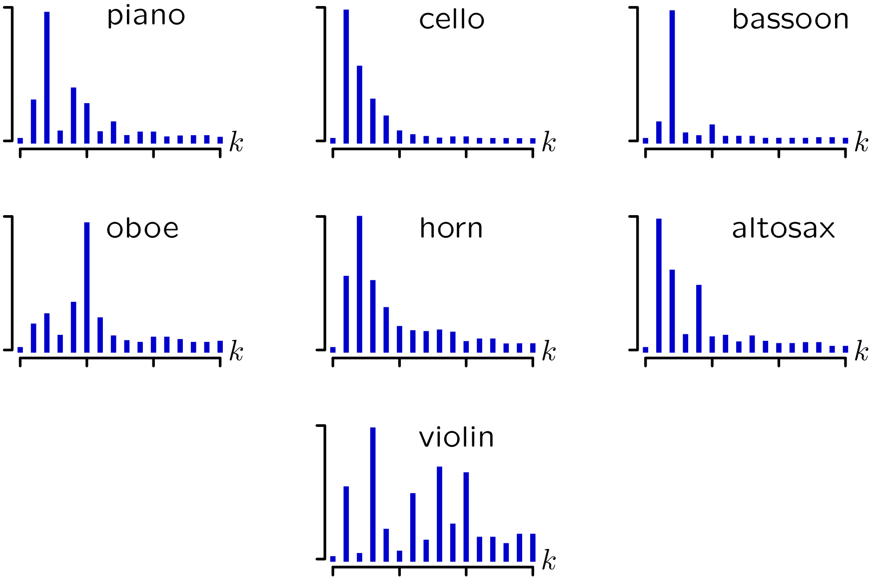

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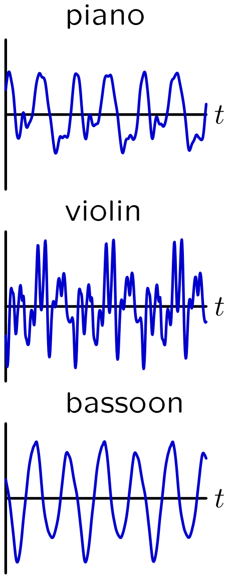
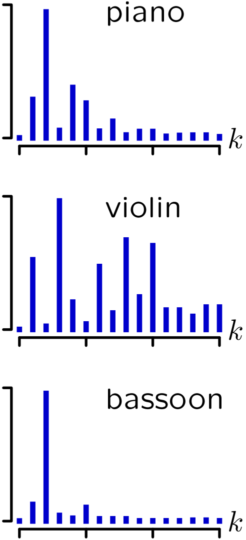
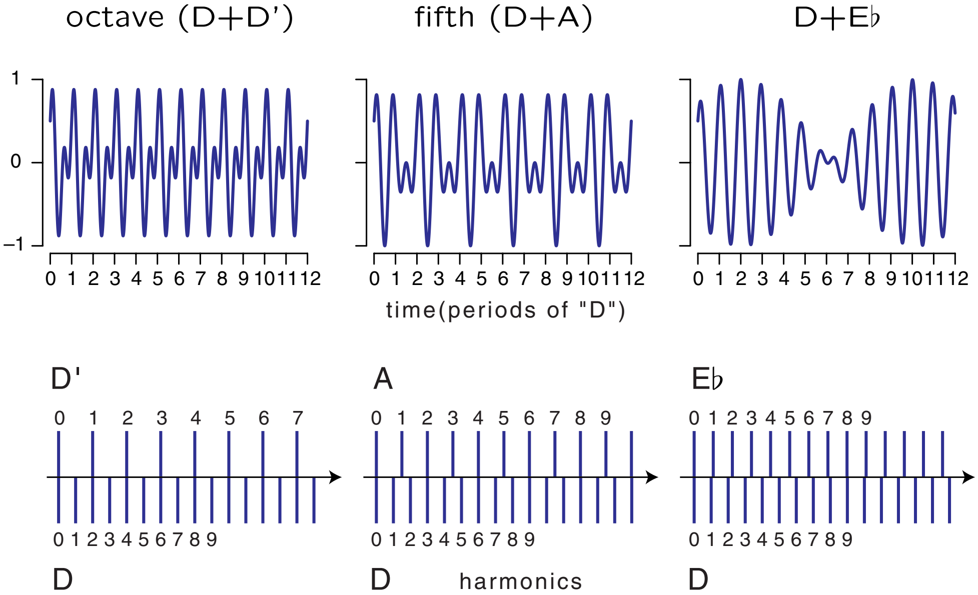




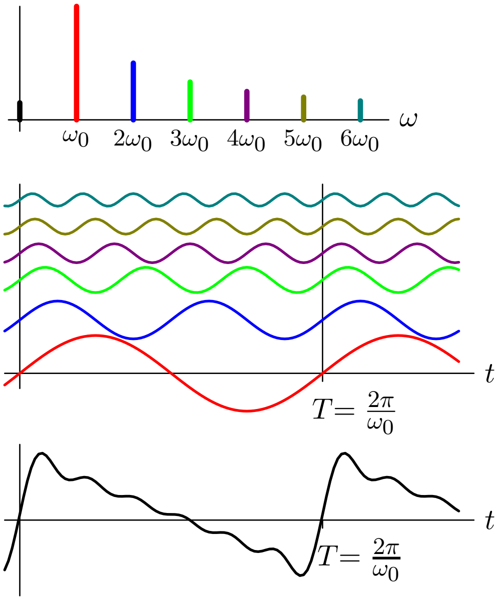
Harmonics

Harmonic structure determines consonance and dissonance.



Harmonic Representations

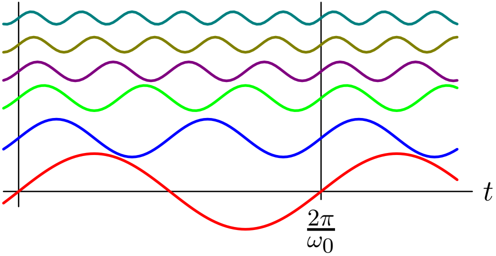
What signals can be represented by sums of harmonic components?

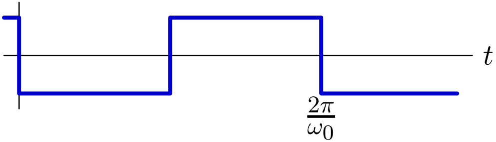


Only periodic signals: all harmonics of ω_0 are periodic in $T = 2\pi/\omega_0$.

Harmonic Representations

Is it possible to represent ALL periodic signals with harmonics? What about discontinuous signals?





Fourier claimed YES — even though all harmonics are continuous!

Lagrange ridiculed the idea that a discontinuous signal could be written as a sum of continuous signals.

We will assume the answer is YES and see if the answer makes sense.

Separating harmonic components

Underlying properties.

1. Multiplying two harmonics produces a new harmonic with the same fundamental frequency:

$$e^{jk\omega_0 t} \times e^{jl\omega_0 t} = e^{j(k+l)\omega_0 t}.$$

2. The integral of a harmonic over any time interval with length equal to a period T is zero unless the harmonic is at DC:

$$\int_{t_0}^{t_0+T} e^{jk\omega_0 t} dt \equiv \int_T e^{jk\omega_0 t} dt = \begin{cases} 0, & k \neq 0 \\ T, & k = 0 \end{cases}$$
$$= T\delta[k]$$

Separating harmonic components

Assume that $x(t)$ is periodic in T and is composed of a weighted sum of harmonics of $\omega_0 = 2\pi/T$.

$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt}$$

Then

$$\begin{aligned}
\int_T x(t)e^{-jl\omega_0 t}dt &= \int_T \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt} e^{-j\omega_0 lt} dt \\
&= \sum_{k=-\infty}^{\infty} a_k \int_T e^{j\omega_0 (k-l)t} dt \\
&= \sum_{k=-\infty}^{\infty} a_k T \delta[k-l] = T a_l
\end{aligned}$$

Therefore

$$a_k = \frac{1}{T}\int_T x(t)e^{-j\omega_0 kt}dt \qquad = \frac{1}{T}\int_T x(t)e^{-j\frac{2\pi}{T}kt}dt$$

Fourier Series

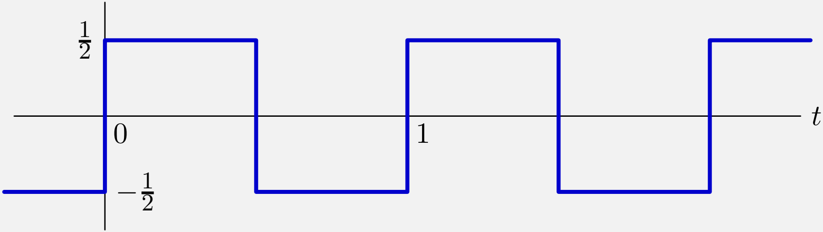
Determining harmonic components of a periodic signal.

$$a_k = \frac{1}{T}\int_T x(t)e^{-j\frac{2\pi}{T}kt}dt$$ (“analysis” equation)

$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt}$$ (“synthesis” equation)

Check Yourself

Let a_k represent the Fourier series coefficients of the following square wave.

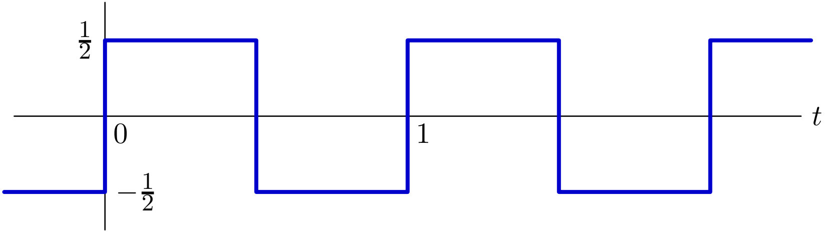


How many of the following statements are true?

1. $a_k = 0$ if k is even
2. a_k is real-valued
3. $|a_k|$ decreases with k^2
4. there are an infinite number of non-zero a_k
5. all of the above

Check Yourself

Let a_k represent the Fourier series coefficients of the following square wave.



$$a_k = \int_T x(t)e^{-j\frac{2\pi}{T}kt}dt = -\frac{1}{2}\int_{-\frac{1}{2}}^{0} e^{-j2\pi kt}dt + \frac{1}{2}\int_{0}^{\frac{1}{2}} e^{-j2\pi kt}dt$$

$$= \frac{1}{j4\pi k}\left(2 - e^{j\pi k} - e^{-j\pi k}\right)$$

$$= \begin{cases} \frac{1}{j\pi k}\ ; & \text{if } k \text{ is odd} \\ 0\ ; & \text{otherwise} \end{cases}$$

Check Yourself

Let a_k represent the Fourier series coefficients of the following square wave.

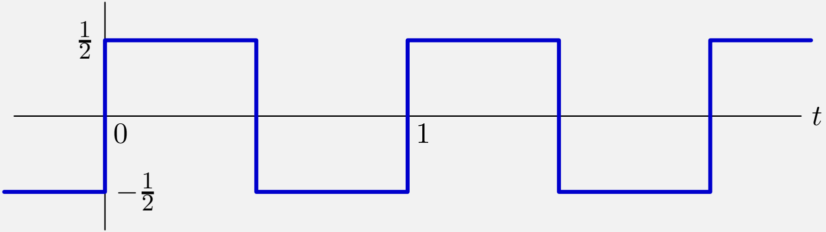
$$a_k = \begin{cases} \dfrac{1}{j\pi k} \,; & \text{if } k \text{ is odd} \\ 0 \,; & \text{otherwise} \end{cases}$$

How many of the following statements are true?

1. $a_k = 0$ if k is even $\surd$
2. a_k is real-valued $\times$
3. $|a_k|$ decreases with k^2 $\times$
4. there are an infinite number of non-zero a_k $\surd$
5. all of the above $\times$

Check Yourself

Let a_k represent the Fourier series coefficients of the following square wave.



How many of the following statements are true? 2

1. $a_k = 0$ if k is even $\surd$
2. a_k is real-valued ✗
3. $|a_k|$ decreases with k^2 ✗
4. there are an infinite number of non-zero a_k $\surd$
5. all of the above ✗

Fourier Series Properties

If a signal is differentiated in time, its Fourier coefficients are multiplied by $j\frac{2\pi}{T}k$.

Proof: Let

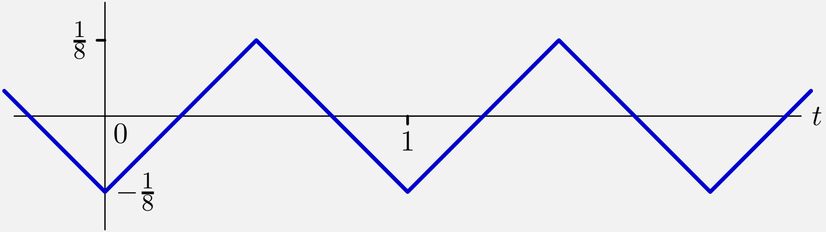
$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt}$$

then

$$\dot{x}(t) = \dot{x}(t+T) = \sum_{k=-\infty}^{\infty} \left(j\frac{2\pi}{T}k\,a_k\right) e^{j\frac{2\pi}{T}kt}$$

Check Yourself

Let b_k represent the Fourier series coefficients of the following triangle wave.

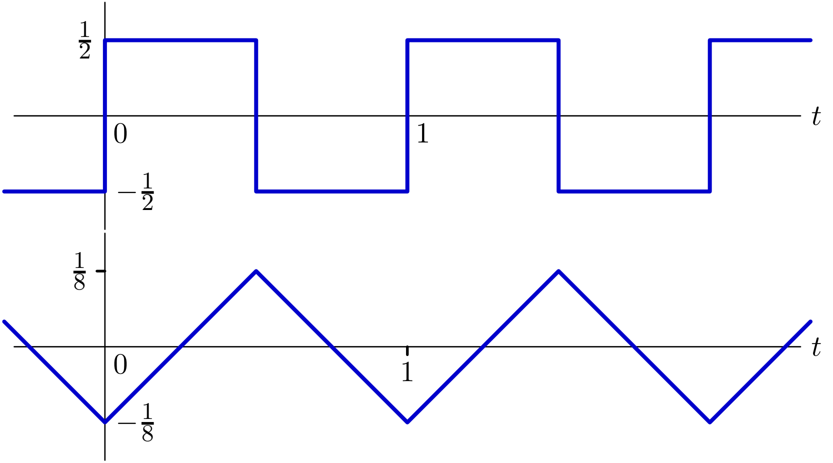


How many of the following statements are true?

1. $b_k = 0$ if k is even
2. b_k is real-valued
3. $|b_k|$ decreases with k^2
4. there are an infinite number of non-zero b_k
5. all of the above

Check Yourself

The triangle waveform is the integral of the square wave.



Therefore the Fourier coefficients of the triangle waveform are $\frac{1}{j2\pi k}$ times those of the square wave.

$$b_k = \frac{1}{jk\pi} \times \frac{1}{j2\pi k} = \frac{-1}{2k^2\pi^2} \;\; ; \;\; k \text{ odd}$$

Check Yourself

Let b_k represent the Fourier series coefficients of the following triangle wave.

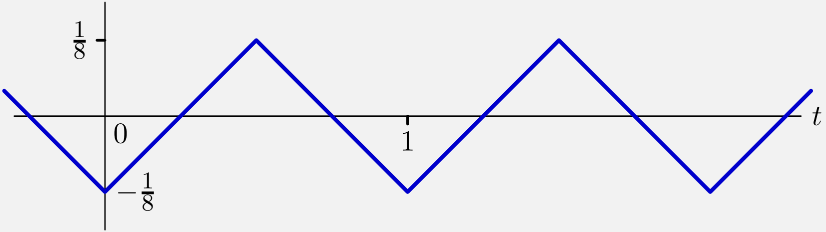
$$b_k = \frac{-1}{2k^2\pi^2} \; ; \; k \text{ odd}$$

How many of the following statements are true?

1. $b_k = 0$ if k is even $\surd$
2. b_k is real-valued $\surd$
3. $|b_k|$ decreases with k^2 $\surd$
4. there are an infinite number of non-zero b_k $\surd$
5. all of the above $\surd$

Check Yourself

Let b_k represent the Fourier series coefficients of the following triangle wave.



How many of the following statements are true? 5

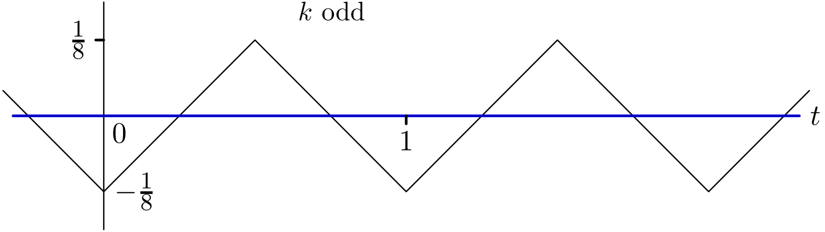
1. $b_k = 0$ if k is even √
2. b_k is real-valued √
3. $|b_k|$ decreases with k^2 √
4. there are an infinite number of non-zero b_k √
5. all of the above √

Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: triangle waveform

$$\sum_{\substack{k=-0 \\ k \text{ odd}}}^{0} \frac{-1}{2k^2\pi^2} e^{j2\pi kt}$$

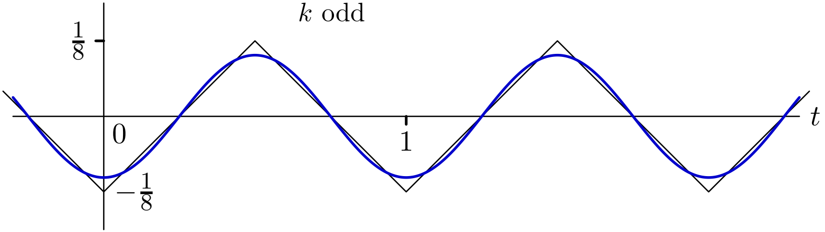


Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: triangle waveform

$$\sum_{\substack{k=-1 \\ k \text{ odd}}}^{1} \frac{-1}{2k^2\pi^2} e^{j2\pi kt}$$

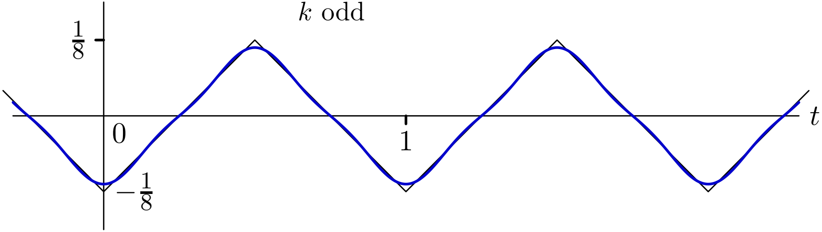


Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: triangle waveform

$$\sum_{\substack{k=-3 \\ k \text{ odd}}}^{3} \frac{-1}{2k^2\pi^2} e^{j2\pi kt}$$

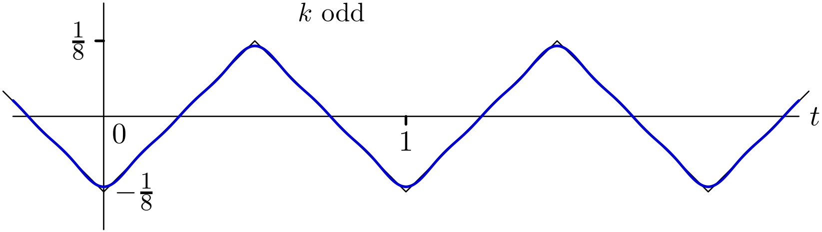


Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: triangle waveform

$$\sum_{\substack{k=-5 \\ k \text{ odd}}}^{5} \frac{-1}{2k^2\pi^2} e^{j2\pi kt}$$

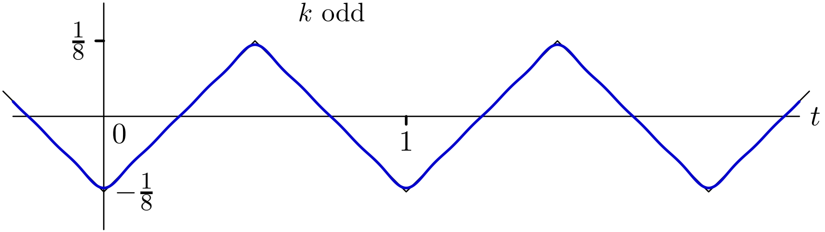


Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: triangle waveform

$$\sum_{\substack{k=-7 \\ k \text{ odd}}}^{7} \frac{-1}{2k^2\pi^2} e^{j2\pi kt}$$

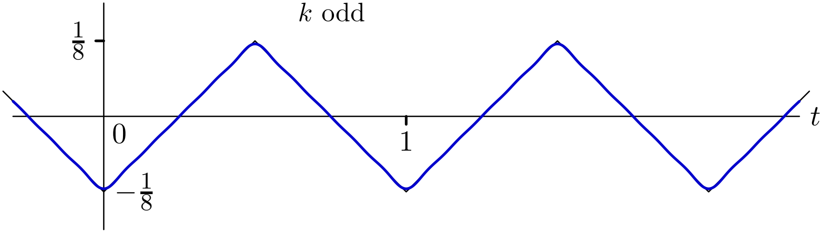


Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: triangle waveform

$$\sum_{\substack{k=-9 \\ k \text{ odd}}}^{9} \frac{-1}{2k^2\pi^2} e^{j2\pi kt}$$

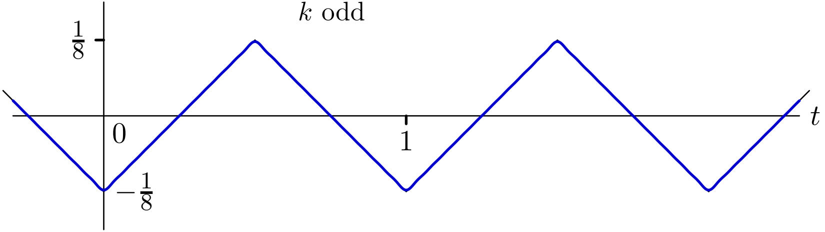


Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: triangle waveform

$$\sum_{\substack{k=-19 \\ k \text{ odd}}}^{19} \frac{-1}{2k^2\pi^2} e^{j2\pi kt}$$

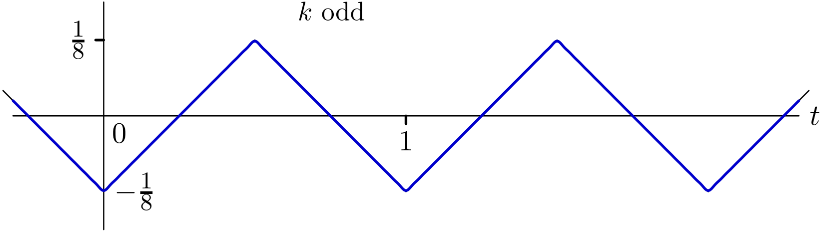


Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: triangle waveform

$$\sum_{\substack{k=-29 \\ k \text{ odd}}}^{29} \frac{-1}{2k^2\pi^2} e^{j2\pi kt}$$

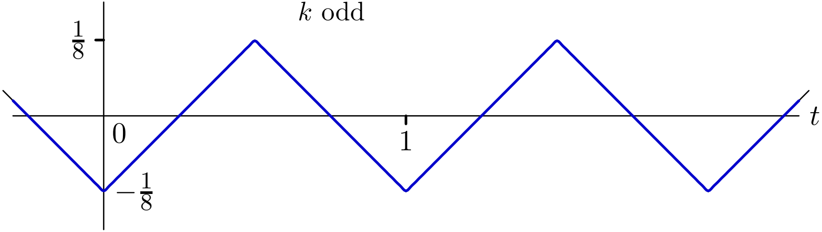


Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: triangle waveform

$$\sum_{\substack{k=-39 \\ k \text{ odd}}}^{39} \frac{-1}{2k^2\pi^2} e^{j2\pi kt}$$



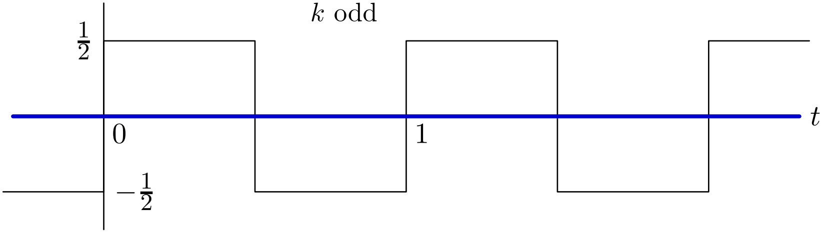
Fourier series representations of functions with discontinuous slopes converge toward functions with discontinuous slopes.

Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: square wave

$$\sum_{\substack{k=-0 \\ k \text{ odd}}}^{0} \frac{1}{jk\pi} e^{j2\pi kt}$$

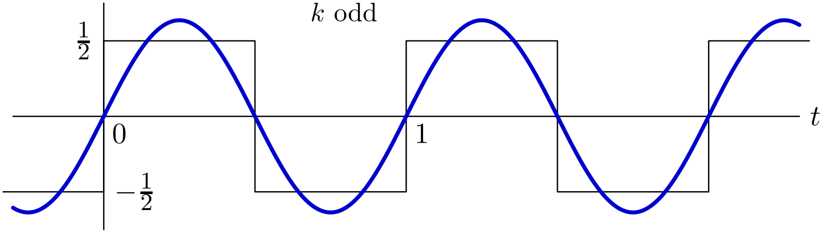


Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: square wave

$$\sum_{\substack{k=-1 \\ k \text{ odd}}}^{1} \frac{1}{jk\pi} e^{j2\pi kt}$$

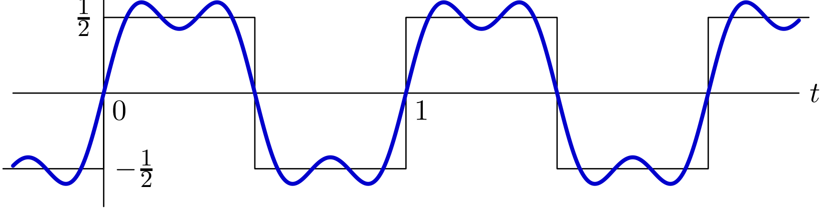


Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: square wave

$$\sum_{\substack{k=-3\\ k \text{ odd}}}^{3} \frac{1}{jk\pi} e^{j2\pi kt}$$

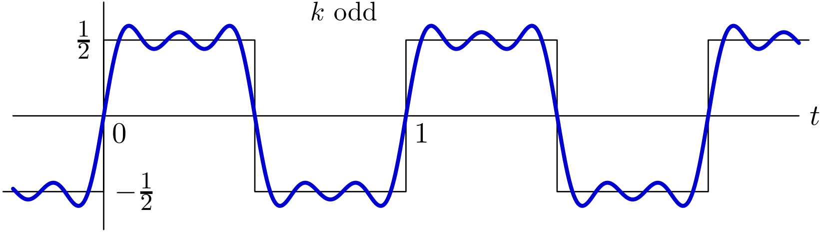


Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: square wave

$$\sum_{\substack{k=-5 \\ k \text{ odd}}}^{5} \frac{1}{jk\pi} e^{j2\pi kt}$$



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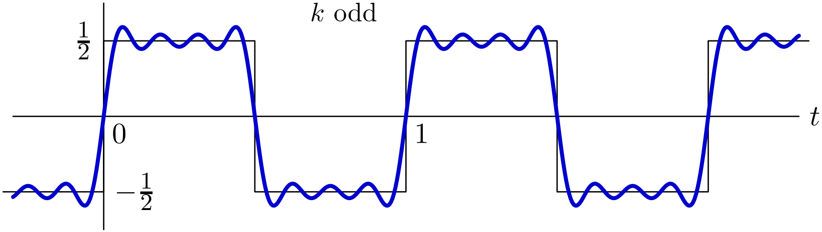
One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: square wave

$$\sum_{\substack{k=-7 \\ k \text{ odd}}}^{7} \frac{1}{jk\pi} e^{j2\pi kt}$$

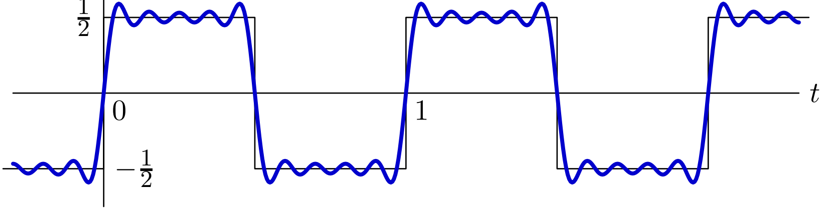


Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: square wave

$$\sum_{\substack{k=-9 \\ k \text{ odd}}}^{9} \frac{1}{jk\pi} e^{j2\pi kt}$$



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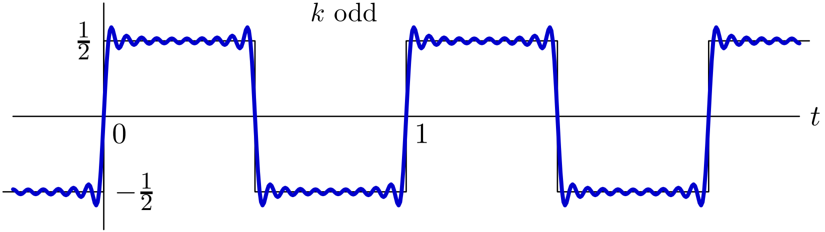
One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: square wave

$$\sum_{\substack{k=-19 \\ k \text{ odd}}}^{19} \frac{1}{jk\pi} e^{j2\pi kt}$$

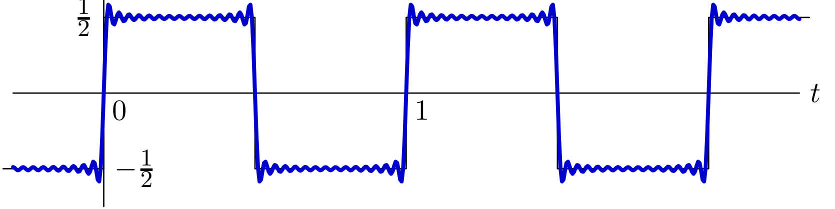


Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: square wave

$$\sum_{\substack{k=-29\\ k \text{ odd}}}^{29} \frac{1}{jk\pi} e^{j2\pi kt}$$

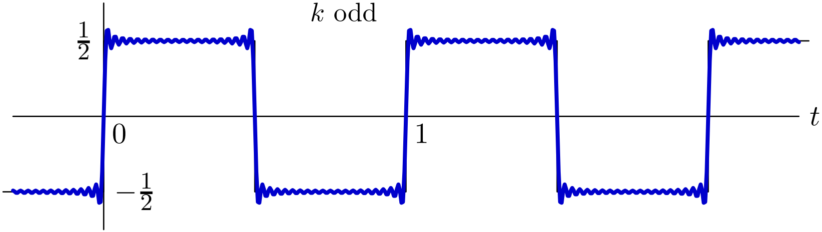


Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

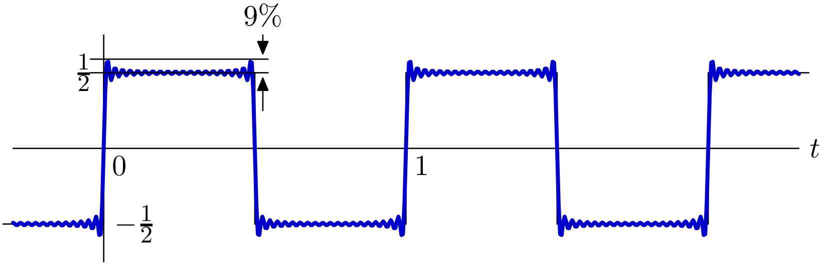
Example: square wave

$$\sum_{\substack{k=-39 \\ k \text{ odd}}}^{39} \frac{1}{jk\pi} e^{j2\pi kt}$$



Fourier Series

Partial sums of Fourier series of discontinuous functions "ring" near discontinuities: Gibb's phenomenon.



This ringing results because the magnitude of the Fourier coefficients is only decreasing as $\frac{1}{k}$ (while they decreased as $\frac{1}{k^2}$ for the triangle).

You can decrease (and even eliminate the ringing) by decreasing the magnitudes of the Fourier coefficients at higher frequencies.

Fourier Series: Summary

Fourier series represent periodic signals as sums of sinusoids.

- valid for an extremely large class of periodic signals
- valid even for discontinuous signals such as square wave

However, convergence as # harmonics increases can be complicated.

Filtering

The output of an LTI system is a "filtered" version of the input.

Input: Fourier series $\rightarrow$ sum of complex exponentials.

$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt}$$

Complex exponentials: eigenfunctions of LTI systems.

$$e^{j\frac{2\pi}{T}kt} \rightarrow H(j\frac{2\pi}{T}k)e^{j\frac{2\pi}{T}kt}$$

Output: same eigenfunctions, amplitudes/phases set by system.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt} \rightarrow y(t) = \sum_{k=-\infty}^{\infty} a_k H(j\frac{2\pi}{T}k)e^{j\frac{2\pi}{T}kt}$$

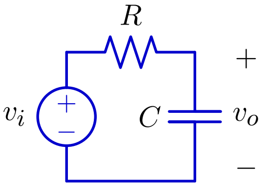
Filtering

Notion of a filter.

LTI systems

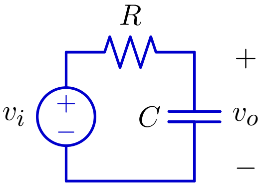
- cannot create new frequencies.
- can scale magnitudes and shift phases of existing components.

Example: Low-Pass Filtering with an RC circuit



Lowpass Filter

Calculate the frequency response of an RC circuit.

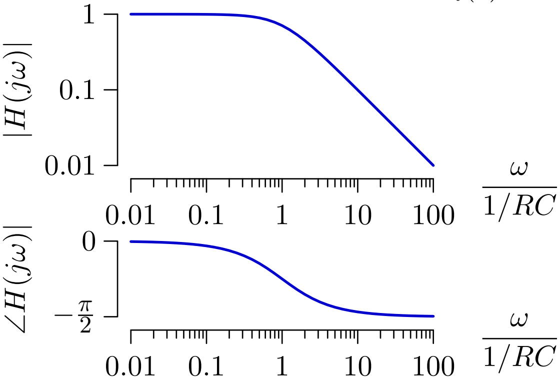


KVL: $v_i(t) = Ri(t) + v_o(t)$

C: $i(t) = C\dot{v}_o(t)$

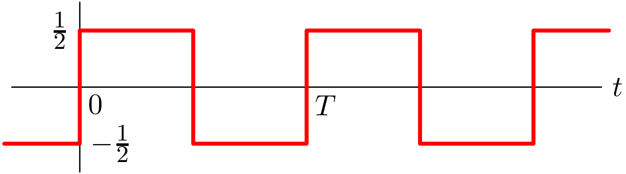
Solving:

$$
\begin{aligned}
v_i(t) &= RC\dot{v}_o(t) + v_o(t) \\
V_i(s) &= (1 + sRC)V_o(s) \\
H(s) &= \frac{V_o(s)}{V_i(s)} = \frac{1}{1 + sRC}
\end{aligned}
$$

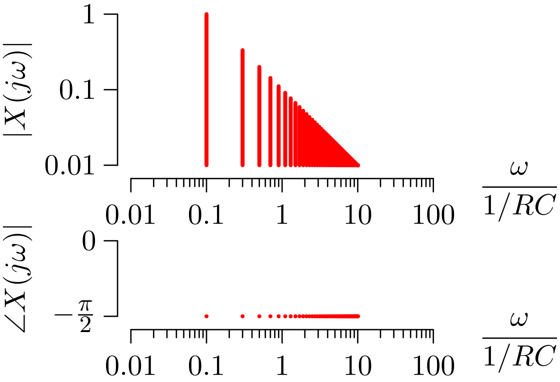


Lowpass Filtering

Let the input be a square wave.

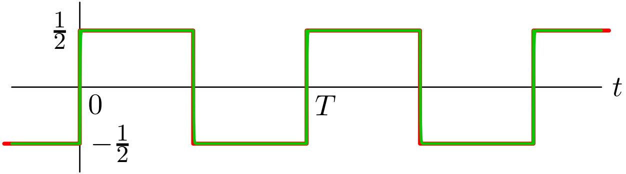


$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 kt} \;; \qquad \omega_0 = \frac{2\pi}{T}$$

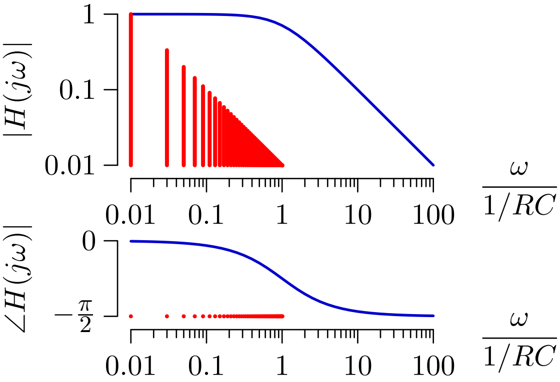


Lowpass Filtering

Low frequency square wave: $\omega_0 << 1/RC$.

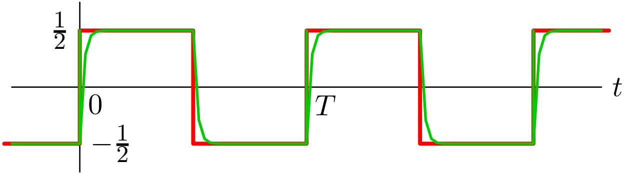


$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 k t} \quad ; \qquad \omega_0 = \frac{2\pi}{T}$$

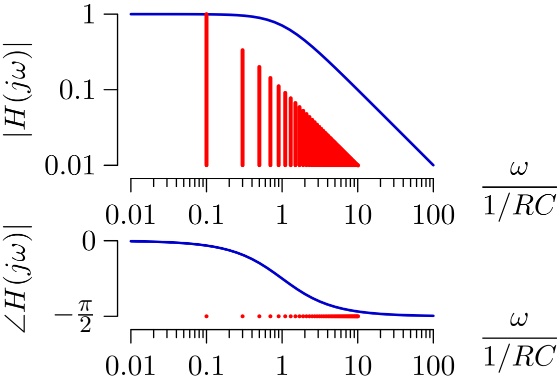


Lowpass Filtering

Higher frequency square wave: $\omega_0 < 1/RC$.

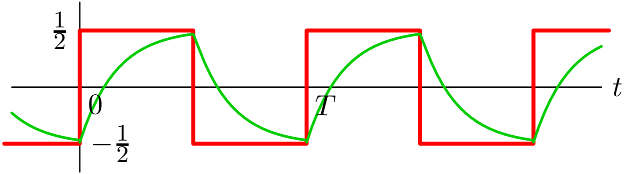


$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 k t} \quad ; \qquad \omega_0 = \frac{2\pi}{T}$$

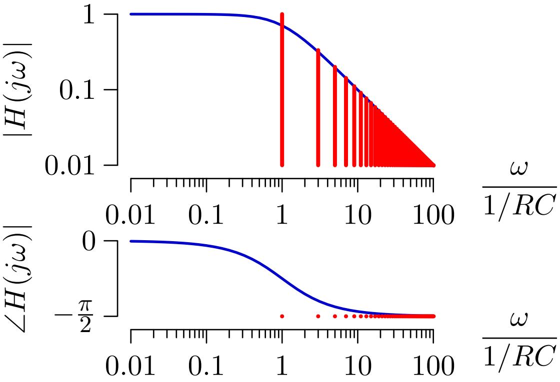


Lowpass Filtering

Still higher frequency square wave: $\omega_0 = 1/RC$.

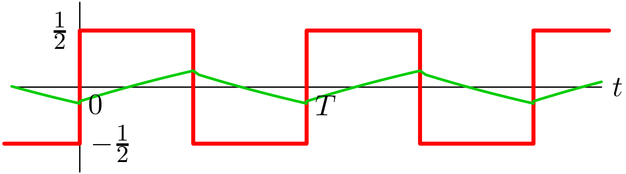


$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 k t} \quad ; \qquad \omega_0 = \frac{2\pi}{T}$$

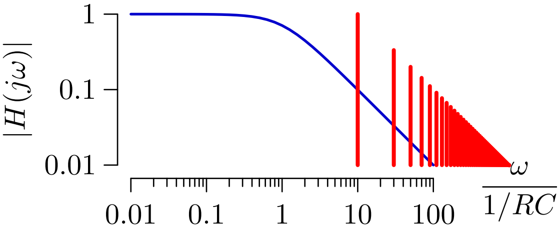


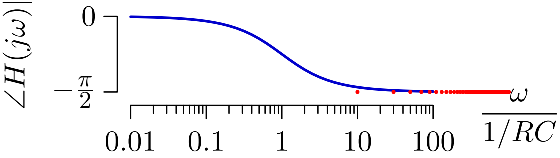
Lowpass Filtering

High frequency square wave: $\omega_0 > 1/RC$.



$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 kt} \quad ; \qquad \omega_0 = \frac{2\pi}{T}$$





Fourier Series: Summary

Fourier series represent signals by their frequency content.

Representing a signal by its frequency content is useful for many signals, e.g., music.

Fourier series motivate a new representation of a system as a filter.