# 6.003: Signals and Systems

## Frequency Response

*October 6, 2011*

## Review

Last time, we saw how a linear, time-invariant (LTI) system can be characterized by its unit-sample/impulse response.

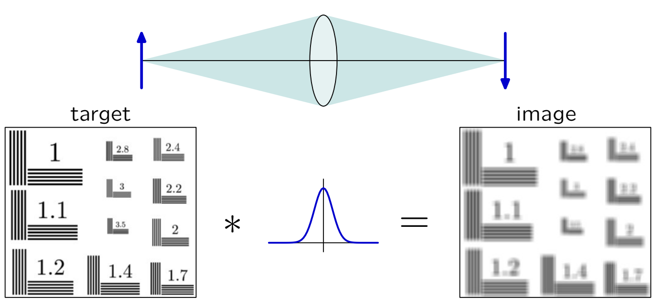
DT: $y[n] = (x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

CT: $y(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$

Characterizing a system by its unit-sample/impulse response is especially insightful for some systems.

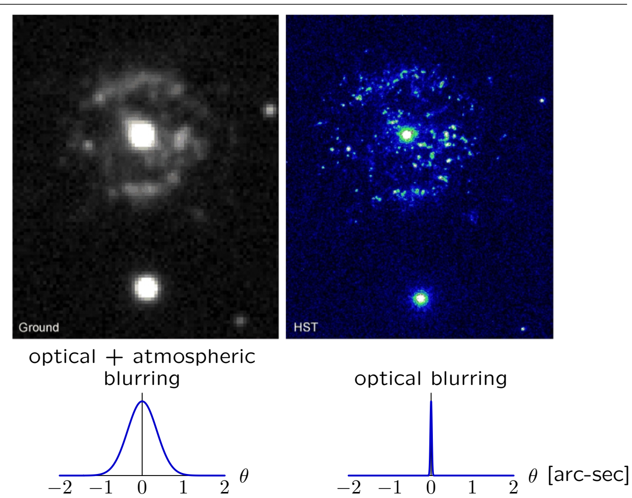
## Microscope

Blurring can be represented by convolving the image with the optical "point-spread-function" (3D impulse response).




Blurring is inversely related to the diameter of the lens.

## Hubble Space Telescope




## Frequency Response

Today we will investigate a different way to characterize a system: the **frequency response**.

Many systems are naturally described by their responses to sinusoids.

Example: audio systems

## Check Yourself

How were frequencies modified in following music clips?

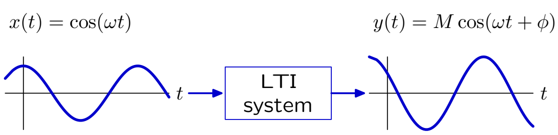
HF: high frequencies — ↑: increased
LF: low frequencies — ↓: decreased

| | clip 1 | clip 2 |
|---|---|---|
| 1. | HF↑ | HF↓ |
| 2. | LF↑ | LF↓ |
| 3. | HF↑ | LF↓ |
| 4. | LF↑ | HF↓ |
| 5. | none of the above | |

## Frequency Response Preview

If the input to a linear, time-invariant system is an eternal sinusoid, then the output is also an eternal sinusoid:

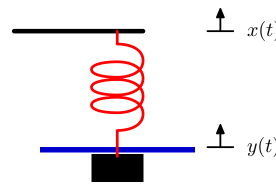
- same frequency
- possibly different amplitude, and
- possibly different phase angle.




The **frequency response** is a plot of the magnitude $M$ and angle $\phi$ as a function of frequency $\omega$.

## Demonstration

Measure the frequency response of a mass, spring, dashpot system.




## Frequency Response

Calculate the frequency response.

Methods

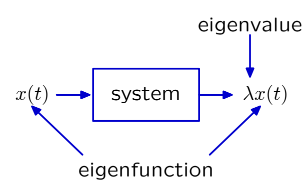
- solve differential equation
  → find particular solution for $x(t) = \cos\omega_0 t$
- find impulse response of system
  → convolve with $x(t) = \cos\omega_0 t$

New method

- use eigenfunctions and eigenvalues

## Eigenfunctions

If the output signal is a scalar multiple of the input signal, we refer to the signal as an eigenfunction and the multiplier as the eigenvalue.




## Check Yourself: Eigenfunctions

Consider the system described by
$\dot{y}(t) + 2y(t) = x(t)$.
Determine if each of the following functions is an eigenfunction of this system. If it is, find its eigenvalue.

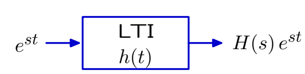
1. $e^{-t}$ for all time
2. $e^{t}$ for all time
3. $e^{jt}$ for all time
4. $\cos(t)$ for all time
5. $u(t)$ for all time

## Complex Exponentials

Complex exponentials are eigenfunctions of LTI systems.

If $x(t) = e^{st}$ and $h(t)$ is the impulse response then

$$y(t) = (h * x)(t) = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau = e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau = H(s)\,e^{st}$$




Eternal sinusoids are sums of complex exponentials.

$$\cos\omega_0 t = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

Furthermore, the eigenvalue associated with $e^{st}$ is $H(s)$!

## Rational System Functions

Eigenvalues are particularly easy to evaluate for systems represented by linear differential equations with constant coefficients.

Then the system function is a ratio of polynomials in $s$.

Example:

$$\ddot{y}(t) + 3\dot{y}(t) + 4y(t) = 2\ddot{x}(t) + 7\dot{x}(t) + 8x(t)$$

Then

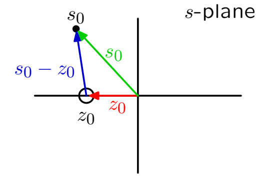
$$H(s) = \frac{2s^2 + 7s + 8}{s^2 + 3s + 4} \equiv \frac{N(s)}{D(s)}$$

## Vector Diagrams

The value of $H(s)$ at a point $s = s_0$ can be determined graphically using vectorial analysis.

Factor the numerator and denominator of the system function to make poles and zeros explicit.

$$H(s_0) = K \frac{(s_0 - z_0)(s_0 - z_1)(s_0 - z_2)\cdots}{(s_0 - p_0)(s_0 - p_1)(s_0 - p_2)\cdots}$$



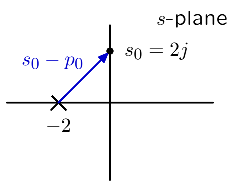

Each factor in the numerator/denominator corresponds to a vector from a zero/pole (here $z_0$) to $s_0$, the point of interest in the $s$-plane.

## Vector Diagrams

Example: Find the response of the system described by

$$H(s) = \frac{1}{s+2}$$

to the input $x(t) = e^{2jt}$ (for all time).




The denominator of $H(s)|_{s=2j}$ is $2j+2$, a vector with length $2\sqrt{2}$ and angle $\pi/4$. Therefore, the response of the system is

$$y(t) = H(2j)e^{2jt} = \frac{1}{2\sqrt{2}} e^{-\frac{j\pi}{4}} e^{2jt}.$$

## Vector Diagrams

The value of $H(s)$ at a point $s = s_0$ can be determined by combining the contributions of the vectors associated with each of the poles and zeros.

$$H(s_0) = K \frac{(s_0 - z_0)(s_0 - z_1)(s_0 - z_2)\cdots}{(s_0 - p_0)(s_0 - p_1)(s_0 - p_2)\cdots}$$

The magnitude is determined by the product of the magnitudes.

$$|H(s_0)| = |K| \frac{|(s_0 - z_0)||(s_0 - z_1)||(s_0 - z_2)|\cdots}{|(s_0 - p_0)||(s_0 - p_1)||(s_0 - p_2)|\cdots}$$

The angle is determined by the sum of the angles.

$$\angle H(s_0) = \angle K + \angle(s_0 - z_0) + \angle(s_0 - z_1) + \cdots - \angle(s_0 - p_0) - \angle(s_0 - p_1) - \cdots$$

## Frequency Response

Response to eternal sinusoids.

Let $x(t) = \cos\omega_0 t$ (for all time). Then

$$x(t) = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

and the response to a sum is the sum of the responses.

$$y(t) = \frac{1}{2}\left(H(j\omega_0)\, e^{j\omega_0 t} + H(-j\omega_0)\, e^{-j\omega_0 t}\right)$$

## Conjugate Symmetry

The complex conjugate of $H(j\omega)$ is $H(-j\omega)$.

The system function is the Laplace transform of the impulse response:

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt$$

where $h(t)$ is a real-valued function of $t$ for physical systems.

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$$

$$H(-j\omega) = \int_{-\infty}^{\infty} h(t)e^{j\omega t}dt \equiv \left(H(j\omega)\right)^*$$

## Frequency Response

Response to eternal sinusoids.

Let $x(t) = \cos\omega_0 t$ (for all time), which can be written as

$$x(t) = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

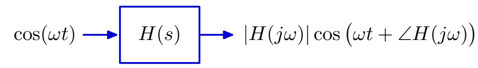
The response to a sum is the sum of the responses,

$$\begin{aligned} y(t) &= \frac{1}{2}\left(H(j\omega_0)e^{j\omega_0 t} + H(-j\omega_0)e^{-j\omega_0 t}\right) \\ &= \text{Re}\left\{H(j\omega_0)e^{j\omega_0 t}\right\} \\ &= \text{Re}\left\{|H(j\omega_0)|e^{j\angle H(j\omega_0)}e^{j\omega_0 t}\right\} \\ &= |H(j\omega_0)|\text{Re}\left\{e^{j\omega_0 t + j\angle H(j\omega_0)}\right\} \end{aligned}$$

$$y(t) = |H(j\omega_0)|\cos\left(\omega_0 t + \angle H(j\omega_0)\right).$$

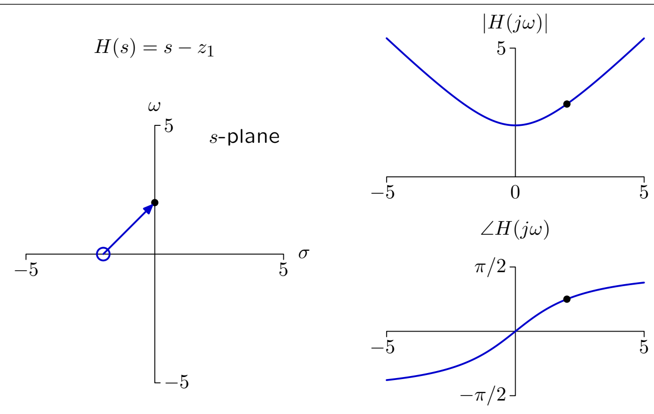
## Frequency Response

The magnitude and phase of the response of a system to an eternal cosine signal is the magnitude and phase of the system function evaluated at $s = j\omega$.



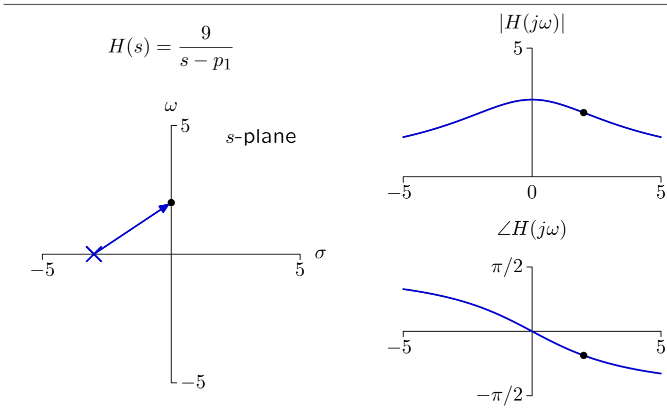

## Vector Diagrams

$$H(s) = s - z_1$$



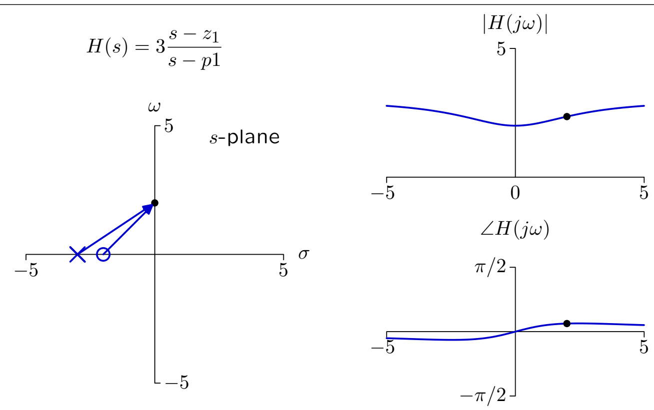

## Vector Diagrams

$$H(s) = \frac{9}{s - p_1}$$

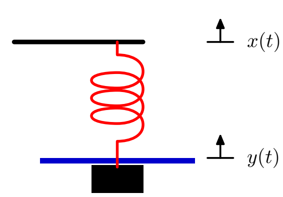



## Vector Diagrams

$$H(s) = 3\frac{s - z_1}{s - p1}$$




## Example: Mass, Spring, and Dashpot




$$F = Ma = M\ddot{y}(t) = K(x(t) - y(t)) - B\dot{y}(t)$$

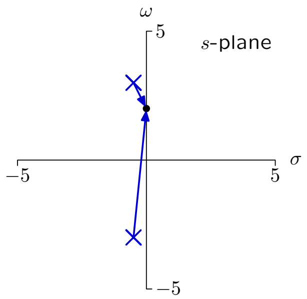
$$M\ddot{y}(t) + B\dot{y}(t) + Ky(t) = Kx(t)$$

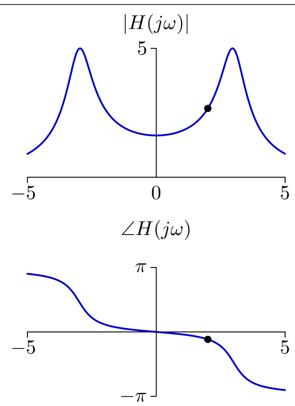
$$\left(s^2M + sB + K\right) Y(s) = KX(s)$$

$$H(s) = \frac{K}{s^2M + sB + K}$$

## Vector Diagrams

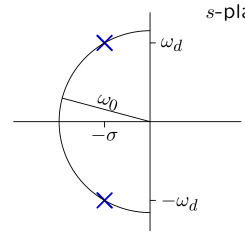
$$H(s) = \frac{15}{(s-p_1)(s-p_2)}$$




## Check Yourself

Consider the system represented by the following poles.

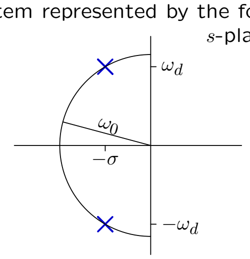



Find the frequency $\omega$ at which the magnitude of the response $y(t)$ is greatest if $x(t) = \cos\omega t$.

1. $\omega = \omega_d$
2. $\omega_d < \omega < \omega_0$
3. $0 < \omega < \omega_d$
4. none of the above

## Check Yourself

Consider the system represented by the following poles.




Find the frequency $\omega$ at which the phase of the response $y(t)$ is $-\pi/2$ if $x(t) = \cos\omega t$.

0. $0 < \omega < \omega_d$
1. $\omega = \omega_d$
2. $\omega_d < \omega < \omega_0$
3. $\omega = \omega_0$
4. $\omega > \omega_0$
5. none

## Frequency Response: Summary

LTI systems can be characterized by responses to eternal sinusoids.

Many systems are naturally described by their frequency response.
– audio systems
– mass, spring, dashpot system

Frequency response is easy to calculate from the system function.

Frequency response lives on the $j\omega$ axis of the Laplace transform.