6.003: Signals and Systems

Discrete Approximation of Continuous-Time Systems

September 29, 2011

Mid-term Examination #1

Wednesday, October 5, 7:30-9:30pm, 26-310, 26-322, 26-328.

No recitations on the day of the exam.

Coverage: CT and DT Systems, Z and Laplace Transforms
Lectures 1–7
Recitations 1–7
Homeworks 1–4

Homework 4 will not collected or graded. Solutions will be posted.

Closed book: 1 page of notes ($8\frac{1}{2} \times 11$ inches; front and back).

No calculators, computers, cell phones, music players, or other aids.

Designed as 1-hour exam; two hours to complete.

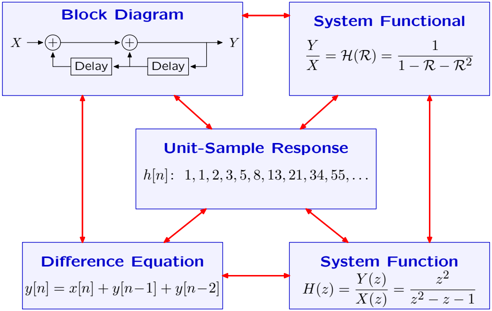
Review sessions during open office hours.

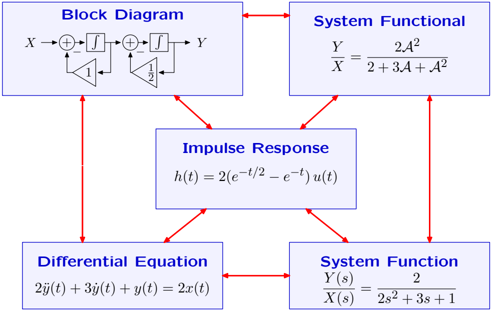
Conflict? Contact freeman@mit.edu before Friday, Sept. 30, 5pm.

Prior term midterm exams have been posted on the 6.003 website.

Concept Map

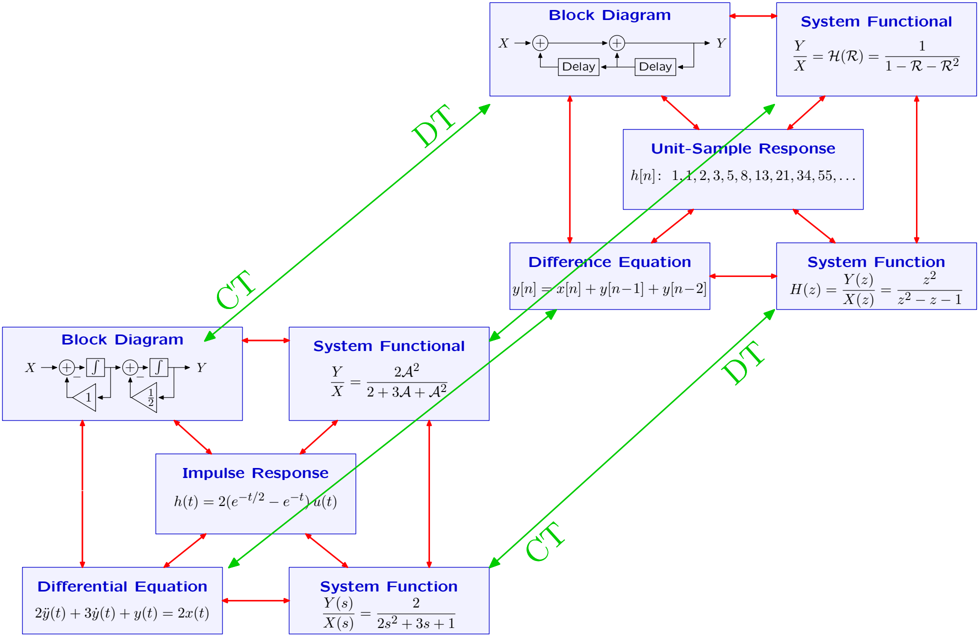
Today we will look at relations between CT and DT representations.





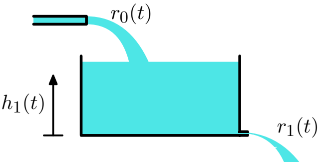
Concept Map

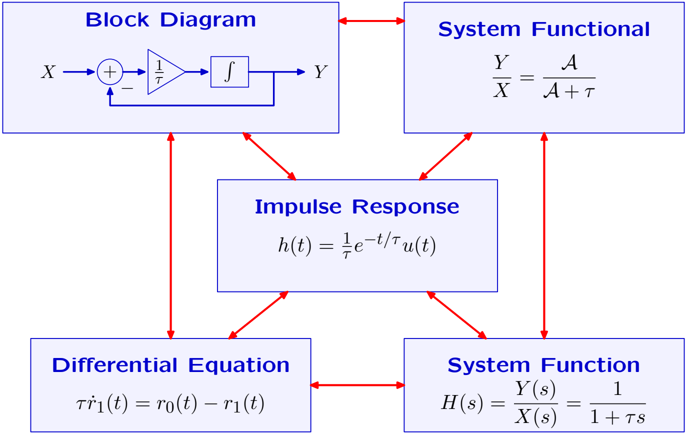
Today we will look at relations between CT and DT representations.



Discrete Approximation of CT Systems

Example: leaky tank

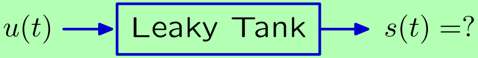


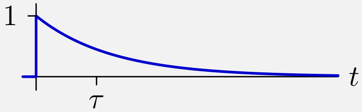


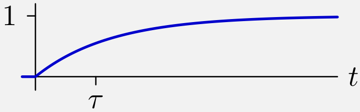
Today: compare step responses of leaky tank and DT approximation.

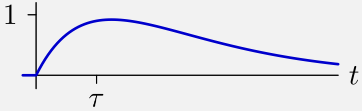
Check Yourself (Practice for Exam)

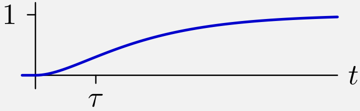
What is the “step response” of the leaky tank system?



1. 

2. 

3. 

4. 

5. none of the above

Check Yourself

What is the "step response" of the leaky tank system?

de: $\tau\,\dot{r}_1(t) = u(t) - r_1(t)$

$t<0$: $r_1(t) = 0$

$t>0$: $r_1(t) = c_1 + c_2 e^{-t/\tau}$

$\dot{r}_1(t) = -\frac{c_2}{\tau} e^{-t/\tau}$

Substitute into de: $\tau\left(-\frac{c_2}{\tau}\right) e^{-t/\tau} = 1 - c_1 - c_2 e^{-t/\tau} \quad \rightarrow \quad c_1 = 1$

Combine $t<0$ and $t>0$:

$$r_1(t) = u(t) + c_2 e^{-t/\tau} u(t)$$

$$\dot{r}_1(t) = \delta(t) + c_2\delta(t) - \frac{c_2}{\tau} e^{-t/\tau} u(t)$$

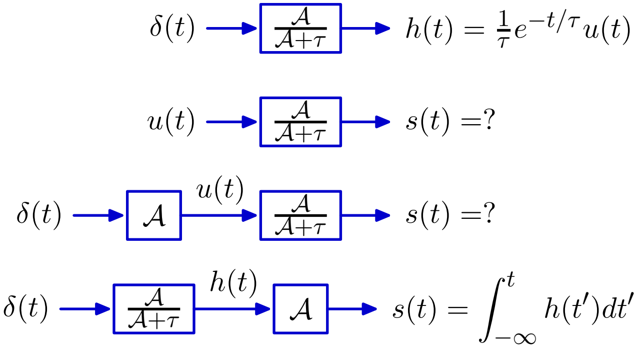
Substitute into de:

$$\tau(1+c_2)\delta(t) - \tau\frac{c_2}{\tau} e^{-t/\tau} u(t) = u(t) - u(t) - c_2 e^{-t/\tau} u(t) \quad \rightarrow \quad c_2 = -1$$

$$r_1(t) = (1 - e^{-t/\tau})u(t)$$

Check Yourself

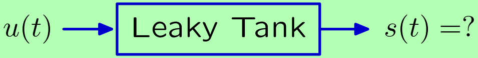
Alternatively, reason with systems!

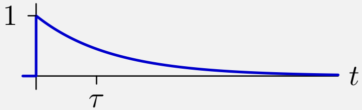


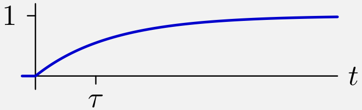
$$s(t) = \int_{-\infty}^{t} \frac{1}{\tau}e^{-t'/\tau}u(t')dt' = \int_{0}^{t} \frac{1}{\tau}e^{-t'/\tau}dt' = \left(1 - e^{-t/\tau}\right)u(t)$$

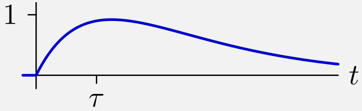
Check Yourself

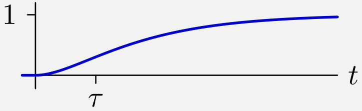
What is the "step response" of the leaky tank system? 2



1. 

2. 

3. 

4. 

5. none of the above

Forward Euler Approximation

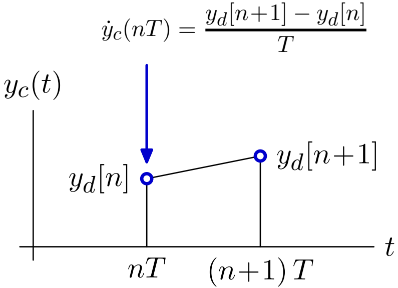
Approximate leaky-tank system using **forward** Euler approach.

Approximate continuous signals by discrete signals:

$$x_d[n] = x_c(nT)$$

$$y_d[n] = y_c(nT)$$

Approximate derivative at $t = nT$ by looking **forward** in time:



Forward Euler Approximation

Approximate leaky-tank system using forward Euler approach.

Substitute

$$x_d[n] = x_c(nT)$$

$$y_d[n] = y_c(nT)$$

$$\dot{y}_c(nT) \approx \frac{y_c\big((n+1)T\big) - y_c\big(nT\big)}{T} = \frac{y_d[n+1] - y_d[n]}{T}$$

into the differential equation

$$\tau \dot{y}_c(t) = x_c(t) - y_c(t)$$

to obtain

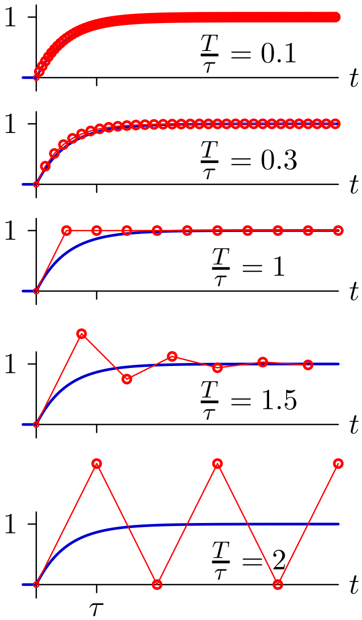
$$\frac{\tau}{T}\Big(y_d[n+1] - y_d[n]\Big) = x_d[n] - y_d[n] \,.$$

Solve:

$$y_d[n+1] - \left(1 - \frac{T}{\tau}\right) y_d[n] = \frac{T}{\tau} x_d[n]$$

Forward Euler Approximation

Plot.



Why is this approximation badly behaved for large $\frac{T}{\tau}$?

Check Yourself

DT approximation:

$$y_d[n+1] - \left(1 - \frac{T}{\tau}\right) y_d[n] = \frac{T}{\tau} x_d[n]$$

Find the DT pole.

1. $z = \dfrac{T}{\tau}$
2. $z = 1 - \dfrac{T}{\tau}$
3. $z = \dfrac{\tau}{T}$
4. $z = -\dfrac{\tau}{T}$
5. $z = \dfrac{1}{1 + \frac{T}{\tau}}$

Check Yourself

DT approximation:

$$y_d[n+1] - \left(1 - \frac{T}{\tau}\right) y_d[n] = \frac{T}{\tau} x_d[n]$$

Take the Z transform:

$$zY_d(z) - \left(1 - \frac{T}{\tau}\right) Y_d(z) = \frac{T}{\tau} X_d(z)$$

Solve for the system function:

$$H(z) = \frac{Y_d(z)}{X_d(z)} = \frac{\frac{T}{\tau}}{z - \left(1 - \frac{T}{\tau}\right)}$$

Pole at $z = 1 - \dfrac{T}{\tau}$.

Check Yourself

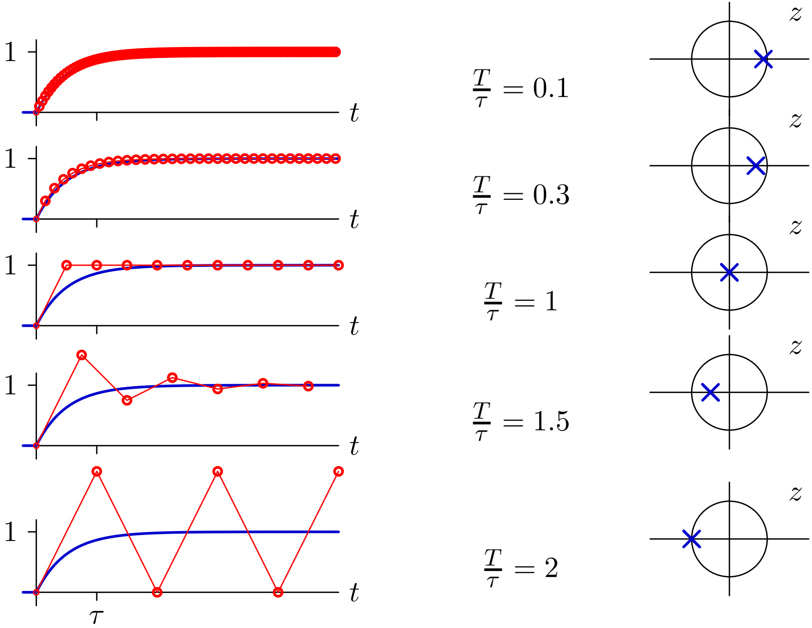
DT approximation:

$$y_d[n+1] - \left(1 - \frac{T}{\tau}\right) y_d[n] = \frac{T}{\tau} x_d[n]$$

Find the DT pole. 2

1. $z = \frac{T}{\tau}$
2. $z = 1 - \frac{T}{\tau}$
3. $z = \frac{\tau}{T}$
4. $z = -\frac{\tau}{T}$
5. $z = \frac{1}{1 + \frac{T}{\tau}}$

Dependence of DT pole on Stepsize



The CT pole was fixed ($s = -\frac{1}{\tau}$). Why is the DT pole changing?

Dependence of DT pole on Stepsize

Dependence of DT pole on T is generic property of forward Euler.

Dependence of DT pole on Stepsize

Dependence of DT pole on T is generic property of forward Euler.

Approach: make a systems model of forward Euler method.

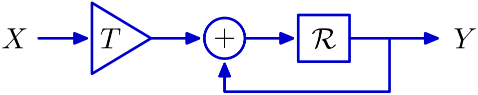
CT block diagrams: adders, gains, and integrators:

$$\dot{y}(t) = x(t)$$

Forward Euler approximation:

$$\frac{y[n+1] - y[n]}{T} = x[n]$$

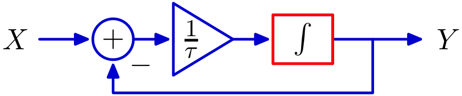
Equivalent system:



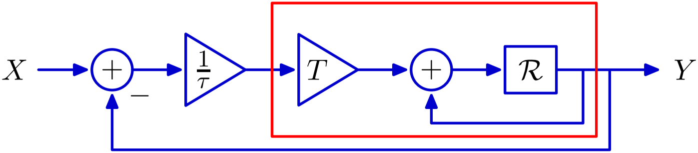
Forward Euler: substitute equivalent system for all integrators.

Example: leaky tank system

Started with leaky tank system:



Replace integrator with forward Euler rule:



Write system functional:

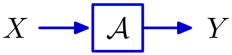
$$\frac{Y}{X} = \frac{\frac{T}{\tau}\frac{\mathcal{R}}{1-\mathcal{R}}}{1+\frac{T}{\tau}\frac{\mathcal{R}}{1-\mathcal{R}}} = \frac{\frac{T}{\tau}\mathcal{R}}{1-\mathcal{R}+\frac{T}{\tau}\mathcal{R}} = \frac{\frac{T}{\tau}\mathcal{R}}{1-\left(1-\frac{T}{\tau}\right)\mathcal{R}}$$

Equivalent to system we previously developed:

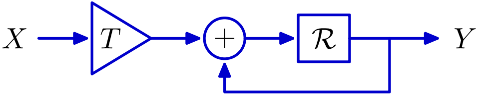
$$y_d[n+1] - \left(1-\frac{T}{\tau}\right)y_d[n] = \frac{T}{\tau}x_d[n]$$

Model of Forward Euler Method

Replace every integrator in the CT system



with the forward Euler model:



Substitute the DT operator for $\mathcal{A}$:

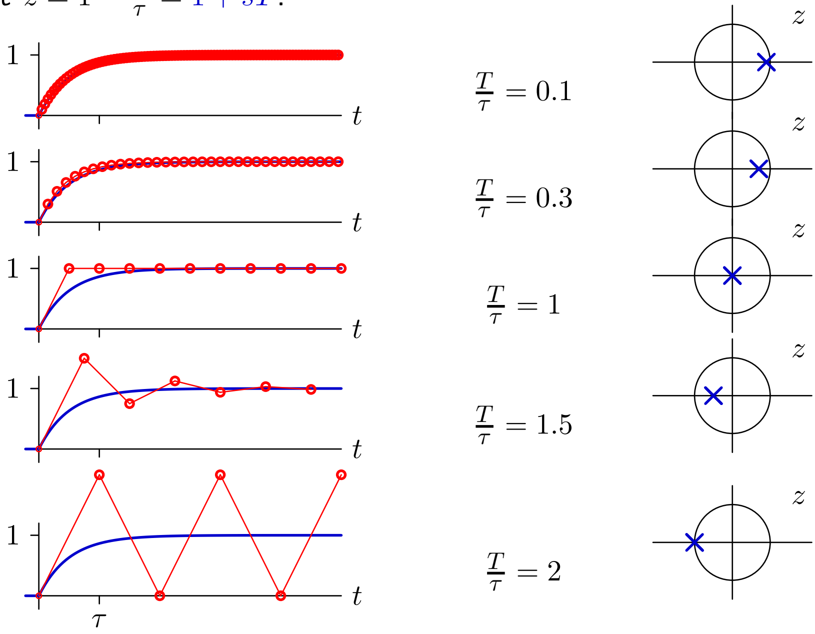
$$\mathcal{A} = \frac{1}{s} \rightarrow \frac{T\mathcal{R}}{1-\mathcal{R}} = \frac{\frac{T}{z}}{1-\frac{1}{z}} = \frac{T}{z-1}$$

Forward Euler maps $s \rightarrow \dfrac{z-1}{T}$.

Or equivalently: $z = 1 + sT$.

Dependence of DT pole on Stepsize

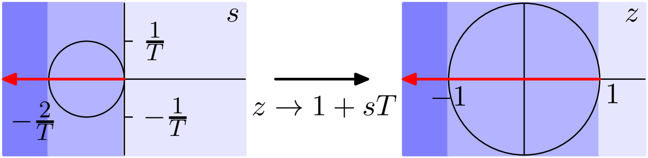
Pole at $z = 1 - \frac{T}{\tau} = 1 + sT$.



Forward Euler: Mapping CT poles to DT poles

Forward Euler Map:

s	$\rightarrow$	$z = 1 + sT$
0		1
$-\frac{1}{T}$		0
$-\frac{2}{T}$		-1



DT stability: CT pole must be inside circle of radius $\frac{1}{T}$ at $s = -\frac{1}{T}$.

$$-\frac{2}{T} < -\frac{1}{\tau} < 0 \quad \rightarrow \quad \frac{T}{\tau} < 2$$

Backward Euler Approximation

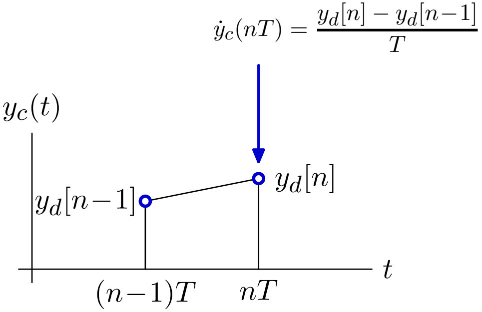
We can do a similar analysis of the **backward** Euler method.

Approximate continuous signals by discrete signals:

$$x_d[n] = x_c(nT)$$

$$y_d[n] = y_c(nT)$$

Approximate derivative at $t = nT$ by looking **backward** in time:



Backward Euler Approximation

We can do a similar analysis of the backward Euler method.

Substitute

$$x_d[n] = x_c(nT)$$

$$y_d[n] = y_c(nT)$$

$$\dot{y}_c(nT) \approx \frac{y_c\big(nT\big) - y_c\big((n-1)T\big)}{T} = \frac{y_d[n] - y_d[n-1]}{T}$$

into the differential equation

$$\tau \dot{y}_c(t) = x_c(t) - y_c(t)$$

to obtain

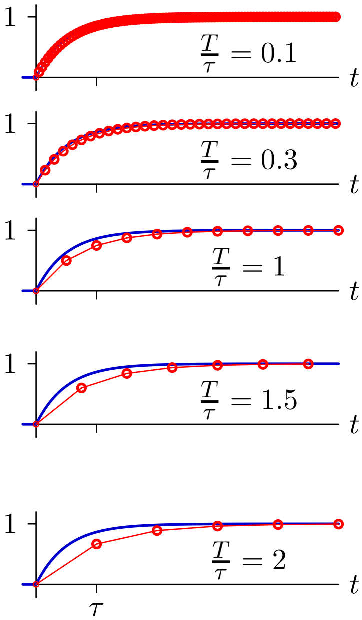
$$\frac{\tau}{T}\Big(y_d[n] - y_d[n-1]\Big) = x_d[n] - y_d[n]\,.$$

Solve:

$$\left(1 + \frac{T}{\tau}\right) y_d[n] - y_d[n-1] = \frac{T}{\tau} x_d[n]$$

Backward Euler Approximation

Plot.



This approximation is better behaved. Why?

Check Yourself

DT approximation:

$$\left(1+\frac{T}{\tau}\right)y_d[n]-y_d[n-1]=\frac{T}{\tau}x_d[n]$$

Find the DT pole.

1. $z=\frac{T}{\tau}$
2. $z=1-\frac{T}{\tau}$
3. $z=\frac{\tau}{T}$
4. $z=-\frac{\tau}{T}$
5. $z=\frac{1}{1+\frac{T}{\tau}}$

Check Yourself

DT approximation:

$$\left(1+\frac{T}{\tau}\right)y_d[n]-y_d[n-1]=\frac{T}{\tau}x_d[n]$$

Take the Z transform:

$$\left(1+\frac{T}{\tau}\right)Y_d(z)-z^{-1}Y_d(z)=\frac{T}{\tau}X_d(z)$$

Find the system function:

$$H(z)=\frac{Y_d(z)}{X_d(z)}=\frac{\frac{T}{\tau}z}{\left(1+\frac{T}{\tau}\right)z-1}$$

Pole at $z=\dfrac{1}{1+\frac{T}{\tau}}$.

Check Yourself

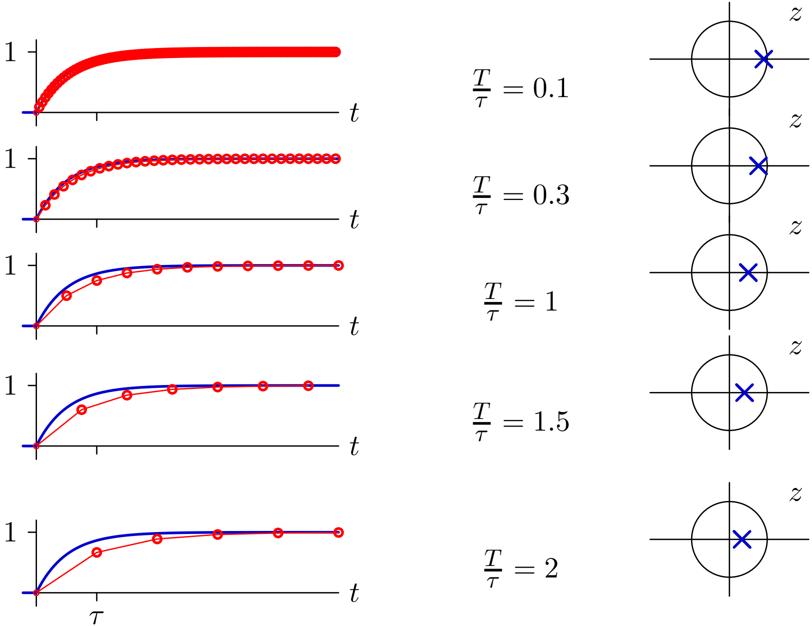
DT approximation:

$$y_d[n+1] - \left(1 - \frac{T}{\tau}\right) y_d[n] = \frac{T}{\tau} x_d[n]$$

Find the DT pole. 5

1. $z = \frac{T}{\tau}$
2. $z = 1 - \frac{T}{\tau}$
3. $z = \frac{\tau}{T}$
4. $z = -\frac{\tau}{T}$
5. $z = \frac{1}{1+\frac{T}{\tau}}$

Dependence of DT pole on Stepsize



$\frac{T}{\tau} = 0.1$

$\frac{T}{\tau} = 0.3$

$\frac{T}{\tau} = 1$

$\frac{T}{\tau} = 1.5$

$\frac{T}{\tau} = 2$

Why is this approximation better behaved?

Dependence of DT pole on Stepsize

Make a systems model of backward Euler method.

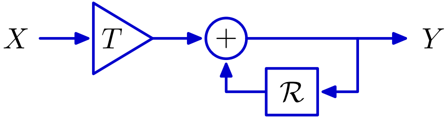
CT block diagrams: adders, gains, and integrators:

$$\dot{y}(t) = x(t)$$

Backward Euler approximation:

$$\frac{y[n] - y[n-1]}{T} = x[n]$$

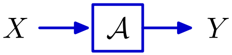
Equivalent system:



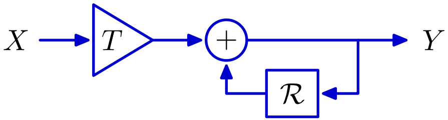
Backward Euler: substitute equivalent system for all integrators.

Model of Backward Euler Method

Replace every integrator in the CT system



with the backward Euler model:



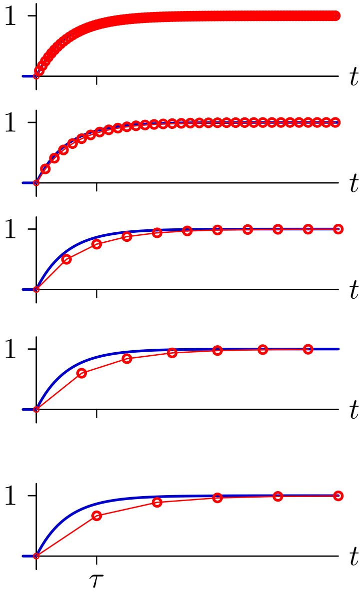
Substitute the DT operator for $\mathcal{A}$:

$$\mathcal{A} = \frac{1}{s} \rightarrow \frac{T}{1-\mathcal{R}} = \frac{T}{1-\frac{1}{z}}$$

Backward Euler maps $z \rightarrow \dfrac{1}{1-sT}$.

Dependence of DT pole on Stepsize

Pole at $z = \frac{1}{1+\frac{T}{\tau}} = \frac{1}{1-sT}$.



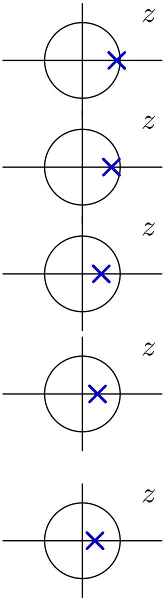
$\frac{T}{\tau} = 0.1$

$\frac{T}{\tau} = 0.3$

$\frac{T}{\tau} = 1$

$\frac{T}{\tau} = 1.5$

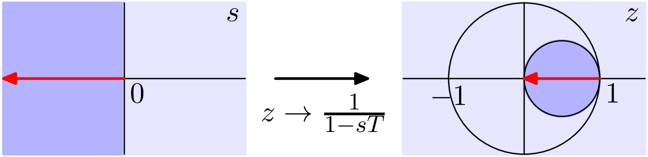
$\frac{T}{\tau} = 2$



Backward Euler: Mapping CT poles to DT poles

Backward Euler Map:

s	$\rightarrow$	$z = \frac{1}{1-sT}$
0		1
$-\frac{1}{T}$		$\frac{1}{2}$
$-\frac{2}{T}$		$\frac{1}{3}$



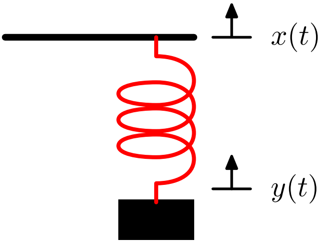
The entire left half-plane maps inside a circle with radius $\frac{1}{2}$ at $z = \frac{1}{2}$.

If CT system is stable, then DT system is also stable.

Masses and Springs, Forwards and Backwards

In Homework 2, you investigated three numerical approximations to a mass and spring system:

- forward Euler
- backward Euler
- centered method



Trapezoidal Rule

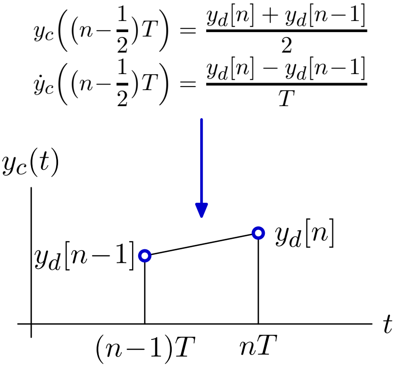
The trapezoidal rule uses centered differences.

Approximate CT signals at points between samples:

$$y_c\Big((n-\frac{1}{2})T\Big) = \frac{y_d[n] + y_d[n-1]}{2}$$

Approximate derivatives at points between samples:

$$\dot{y}_c\Big((n-\frac{1}{2})T\Big) = \frac{y_d[n] - y_d[n-1]}{T}$$



Trapezoidal Rule

The trapezoidal rule uses centered differences.

$$\dot{y}(t) = x(t)$$

Trapezoidal rule:

$$\frac{y[n] - y[n-1]}{T} = \frac{x[n] + x[n-1]}{2}$$

Z transform:

$$H(z) = \frac{Y(s)}{X(s)} = \frac{T}{2}\left(\frac{1 + z^{-1}}{1 - z^{-1}}\right) = \frac{T}{2}\left(\frac{z+1}{z-1}\right)$$

Map:

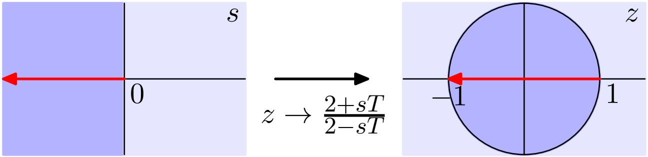
$$\mathcal{A} = \frac{1}{s} \rightarrow \frac{T}{2}\left(\frac{z+1}{z-1}\right)$$

Trapezoidal rule maps $z \rightarrow \dfrac{1 + \frac{sT}{2}}{1 - \frac{sT}{2}}$.

Trapezoidal Rule: Mapping CT poles to DT poles

Trapezoidal Map:

s	$\rightarrow$	$z = \frac{1+\frac{sT}{2}}{1-\frac{sT}{2}}$
0		1
$-\frac{1}{T}$		$\frac{1}{3}$
$-\frac{2}{T}$		0
$-\infty$		-1
$j\omega$		$\frac{2+j\omega T}{2-j\omega T}$

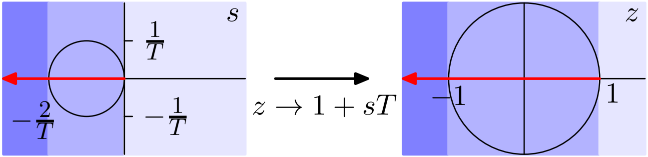


The entire left-half plane maps inside the unit circle.

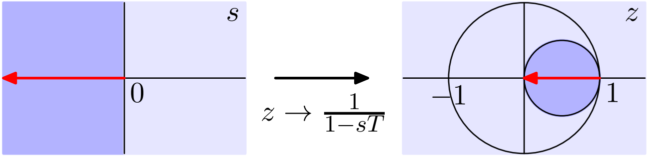
The $j\omega$ axis maps onto the unit circle

Mapping s to z: Leaky-Tank System

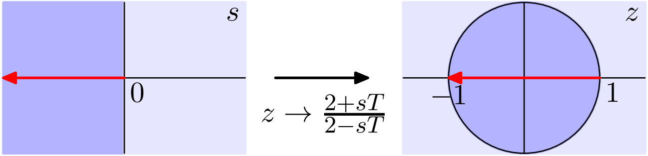
Forward Euler Method



Backward Euler Method

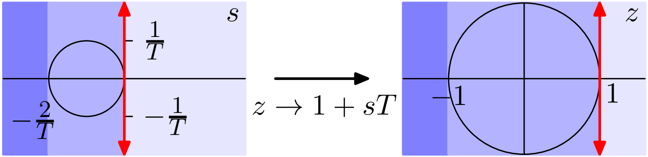


Trapezoidal Rule

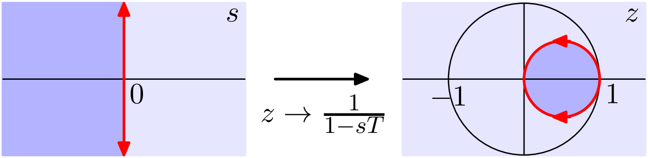


Mapping s to z: Mass and Spring System

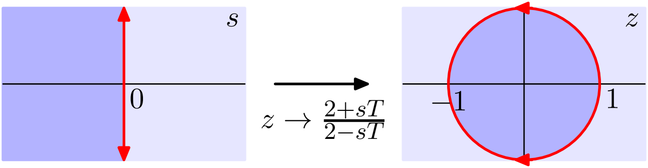
Forward Euler Method



Backward Euler Method

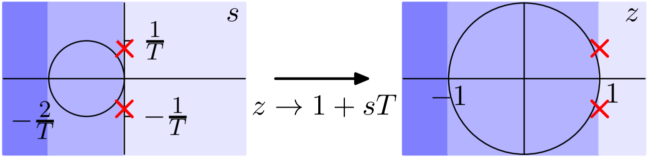


Trapezoidal Rule

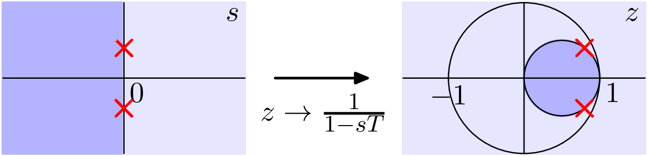


Mapping s to z: Mass and Spring System

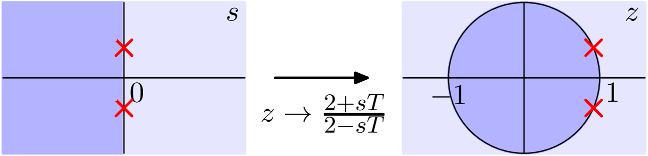
Forward Euler Method



Backward Euler Method



Trapezoidal Rule



Concept Map

Relations between CT and DT representations.