6.003: Signals and Systems

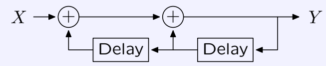
Z Transform

September 22, 2011

Concept Map: Discrete-Time Systems

Multiple representations of DT systems.

Block Diagram



System Functional

$$\frac{Y}{X} = \mathcal{H}(\mathcal{R}) = \frac{1}{1-\mathcal{R}-\mathcal{R}^2}$$

Unit-Sample Response

$h[n]:\ 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$

Difference Equation

$y[n] = x[n] + y[n-1] + y[n-2]$

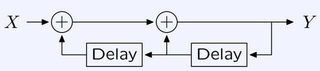
System Function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^2}{z^2-z-1}$$

Concept Map: Discrete-Time Systems

Relations among representations.

Block Diagram



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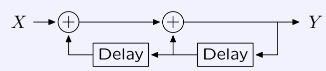
Concept Map: Discrete-Time Systems

Two interpretations of "Delay."

Delay $\to \mathcal{R}$

Block Diagram



System Functional

$$\frac{Y}{X} = \mathcal{H}(\mathcal{R}) = \frac{1}{1-\mathcal{R}-\mathcal{R}^2}$$

index shift

Unit-Sample Response

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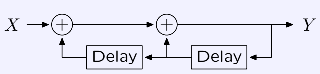
Concept Map: Discrete-Time Systems

Relation between System Functional and System Function.

Block Diagram



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Unit-Sample Response

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$\mathcal{R} \to \frac{1}{z}$

Difference Equation

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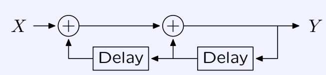
System Function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^2}{z^2-z-1}$$

Check Yourself

What is relation of System Functional to Unit-Sample Response

Block Diagram



System Functional

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Check Yourself

What is relation of System Function to Unit-Sample Response?

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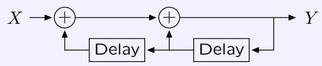

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Z Transform

We call the relation between $H(z)$ and $h[n]$ the Z transform.

$$H(z) = \sum_n h[n]z^{-n}$$

Z transform maps a function of discrete time n to a function of z.

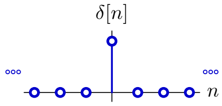
Although motivated by system functions, we can define a Z transform for any signal.

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Notice that we include $n < 0$ as well as $n > 0 \rightarrow$ **bilateral** Z transform (there is also a unilateral Z transform with similar but not identical properties).

Simple Z transforms

Find the Z transform of the unit-sample signal.

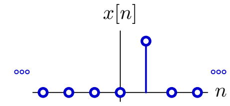


$x[n] = \delta[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = x[0]z^0 = 1$$

Simple Z transforms

Find the Z transform of a delayed unit-sample signal.



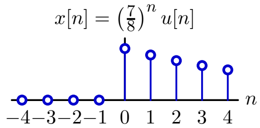
$x[n] = \delta[n-1]$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = x[1]z^{-1} = z^{-1}$$

Check Yourself

What is the Z transform of the following signal.

$x[n] = \left(\frac{7}{8}\right)^n u[n]$



1. $\dfrac{1}{1-\frac{7}{8}z}$
2. $\dfrac{1}{1-\frac{7}{8}z^{-1}}$
3. $\dfrac{z}{1-\frac{7}{8}z}$
4. $\dfrac{z^{-1}}{1-\frac{7}{8}z^{-1}}$
5. none

Z Transform Pairs

The signal $x[n]$, which is a function of time n, maps to a Z transform $X(z)$, which is a function of z.

$$x[n] = \left(\frac{7}{8}\right)^n u[n] \quad \leftrightarrow \quad X(z) = \frac{1}{1-\frac{7}{8}z^{-1}}$$

For what values of z does $X(z)$ make sense?

Z Transform Pairs

The signal $x[n]$, which is a function of time n, maps to a Z transform $X(z)$, which is a function of z.

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For what values of z does $X(z)$ make sense?

The Z transform is only defined for values of z for which the defining sum converges.

$$X(z) = \sum_{n=-\infty}^{\infty} \left(\frac{7}{8}\right)^n z^{-n} u[n] = \sum_{n=0}^{\infty} \left(\frac{7}{8}\right)^n z^{-n} = \frac{1}{1-\frac{7}{8}z^{-1}}$$

Therefore $\left|\frac{7}{8}z^{-1}\right| < 1$, i.e., $|z| > \frac{7}{8}$.

Regions of Convergence

The Z transform $X(z)$ is a function of z defined for all z inside a **Region of Convergence (ROC)**.

$$x[n] = \left(\frac{7}{8}\right)^n u[n] \quad \leftrightarrow \quad X(z) = \frac{1}{1-\frac{7}{8}z^{-1}}\,; \qquad |z| > \frac{7}{8}$$

ROC: $|z| > \frac{7}{8}$

Z Transform Mathematics

Based on **properties** of the Z transform.

Linearity:

if	$x_1[n]$	$\leftrightarrow$	$X_1(z)$	for z in ROC$_1$
and	$x_2[n]$	$\leftrightarrow$	$X_2(z)$	for z in ROC$_2$
then	$x_1[n]+x_2[n]$	$\leftrightarrow$	$X_1(z)+X_2(z)$	for z in (ROC$_1 \cap$ ROC$_2$).

Let $y[n] = x_1[n] + x_2[n]$ then

$$\begin{aligned} Y(z) &= \sum_{n=-\infty}^{\infty} y[n]z^{-n} \\ &= \sum_{n=-\infty}^{\infty} (x_1[n]+x_2[n])z^{-n} \\ &= \sum_{n=-\infty}^{\infty} x_1[n]z^{-n} + \sum_{n=-\infty}^{\infty} x_2[n]z^{-n} \\ &= X_1(z) + X_2(z) \end{aligned}$$

Delay Property

If $x[n] \leftrightarrow X(z)$ for z in ROC then $x[n-1] \leftrightarrow z^{-1}X(z)$ for z in ROC.

We have already seen an example of this property.

$$\delta[n] \leftrightarrow 1$$
$$\delta[n-1] \leftrightarrow z^{-1}$$

More generally,

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Let $y[n] = x[n-1]$ then

$$Y(z) = \sum_{n=-\infty}^{\infty} y[n]z^{-n} = \sum_{n=-\infty}^{\infty} x[n-1]z^{-n}$$

Substitute $m = n-1$

$$Y(z) = \sum_{m=-\infty}^{\infty} x[m]z^{-m-1} = z^{-1}X(z)$$

Rational Polynomials

A system that can be described by a linear difference equation with constant coefficients can also be described by a Z transform that is a **ratio of polynomials** in z.

$$b_0y[n]+b_1y[n-1]+b_2y[n-2]+\cdots = a_0x[n]+a_1x[n-1]+a_2x[n-2]+\cdots$$

Taking the Z transform of both sides, and applying the delay property

$$b_0Y(z)+b_1z^{-1}Y(z)+b_2z^{-2}Y(z)+\cdots = a_0X(z)+a_1z^{-1}X(z)+a_2z^{-2}X(z)+\cdots$$

$$\begin{aligned} H(z) = \frac{Y(z)}{X(z)} &= \frac{a_0+a_1z^{-1}+a_2z^{-2}+\cdots}{b_0+b_1z^{-1}+b_2z^{-2}+\cdots} \\ &= \frac{a_0z^k+a_1z^{k-1}+a_2z^{k-2}+\cdots}{b_0z^k+b_1z^{k-1}+b_2z^{k-2}+\cdots} \end{aligned}$$

Rational Polynomials

Applying the fundamental theorem of algebra and the factor theorem, we can express the polynomials as a product of factors.

$$\begin{aligned} H(z) &= \frac{a_0z^k+a_1z^{k-1}+a_2z^{k-2}+\cdots}{b_0z^k+b_1z^{k-1}+b_2z^{k-2}+\cdots} \\ &= \frac{(z-z_0)\ (z-z_1)\ \cdots\ (z-z_k)}{(z-p_0)\ (z-p_1)\ \cdots\ (z-p_k)} \end{aligned}$$

where the roots are called **poles** and **zeros**.

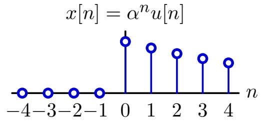
Rational Polynomials

Regions of convergence for Z transform are delimited by circles in the Z-plane. The edges of the circles are at the poles.

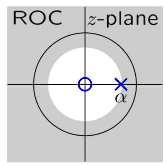
Example: $x[n] = \alpha^n u[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} \alpha^n u[n] z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n}$$
$$= \frac{1}{1-\alpha z^{-1}}\,; \quad \left|\alpha z^{-1}\right| < 1$$
$$= \frac{z}{z-\alpha}\,; \quad |z| > |\alpha|$$

$x[n] = \alpha^n u[n]$



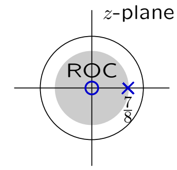
$\frac{z}{z-\alpha}$



Check Yourself

What DT signal has the following Z transform?

$\frac{z}{z-\frac{7}{8}}\,; \quad |z| < \frac{7}{8}$



Check Yourself

Find the inverse transform of

$$X(z) = \frac{-3z}{2z^2 - 5z + 2}$$

given that the ROC includes the unit circle.

Solving Difference Equations with Z Transforms

Start with difference equation:

$$y[n] - \frac{1}{2} y[n-1] = \delta[n]$$

Take the Z transform of this equation:

$$Y(z) - \frac{1}{2} z^{-1} Y(z) = 1$$

Solve for $Y(z)$:

$$Y(z) = \frac{1}{1 - \frac{1}{2} z^{-1}}$$

Take the inverse Z transform (by recognizing the form of the transform):

$$y[n] = \left(\frac{1}{2}\right)^n u[n]$$

Inverse Z transform

The inverse Z transform is defined by an integral that is not particularly easy to solve.

Formally,

$$x[n] = \frac{1}{2\pi j} \int_C X(z) z^{n-1} dz$$

where C represents a closed contour that circles the origin by running in a counterclockwise direction through the region of convergence. This integral is not generally easy to compute.

This equation can be useful to prove theorems.

There are better ways (e.g., partial fractions) to compute inverse transforms for the kinds of systems that we frequently encounter.

Properties of Z Transforms

The use of Z Transforms to solve differential equations depends on several important properties.

Property	$x[n]$	$X(z)$	ROC
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	$\supset (R_1 \cap R_2)$
Delay	$x[n-1]$	$z^{-1}X(z)$	R
Multiply by n	$nx[n]$	$-z\frac{dX(z)}{dz}$	R
Convolve in n	$\sum_{m=-\infty}^{\infty} x_1[m]x_2[n-m]$	$X_1(z)X_2(z)$	$\supset (R_1 \cap R_2)$

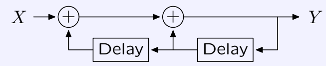
Check Yourself

Find the inverse transform of $Y(z) = \left(\frac{z}{z-1}\right)^2$; $|z| > 1$.

Concept Map: Discrete-Time Systems

Relations among representations.

Block Diagram



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