6.003: Signals and Systems

Feedback, Poles, and Fundamental Modes

September 15, 2011

Homework

Doing the homework is essential to understanding the content.

Weekly Homework Assigments

- tutor (exam-type) problems:
  answers are automatically checked to provide quick feedback
- engineering design (real-world) problems:
  graded by a human

Learning doesn't end when you have submitted your work!

- solutions will be posted on Wednesdays at 5pm
- read solutions to find errors and to see alternative approaches
- mark the errors in your previously submitted work
- submit the markup by Friday at 5pm
- **identify ALL errors and get back half of the points you lost!**

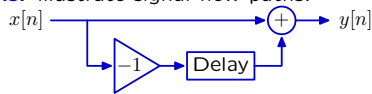## Last Time: Multiple Representations of DT Systems

Verbal descriptions: preserve the rationale.

"To reduce the number of bits needed to store a sequence of large numbers that are nearly equal, record the first number, and then record successive differences."

Difference equations: mathematically compact.

$y[n] = x[n] - x[n-1]$

Block diagrams: illustrate signal flow paths.

Operator representations: analyze systems as polynomials.
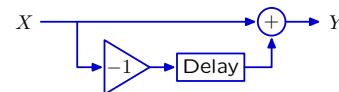
$Y = (1 - \mathcal{R})\, X$

Last Time: Feedback, Cyclic Signal Paths, and Modes

Systems with signals that depend on previous values of the same signal are said to have **feedback**.
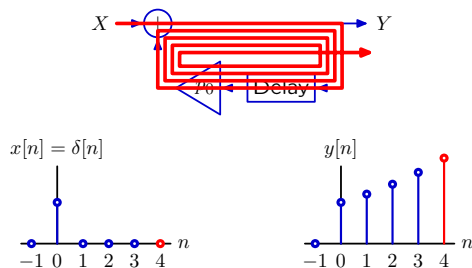
Example: The accumulator system has feedback.

By contrast, the difference machine does not have feedback.


Last Time: Feedback, Cyclic Signal Paths, and Modes

The effect of feedback can be visualized by tracing each cycle through the cyclic signal paths.

$x[n] = \delta[n]$

$y[n]$

Each cycle creates another sample in the output.

The response will persist even though the input is transient.
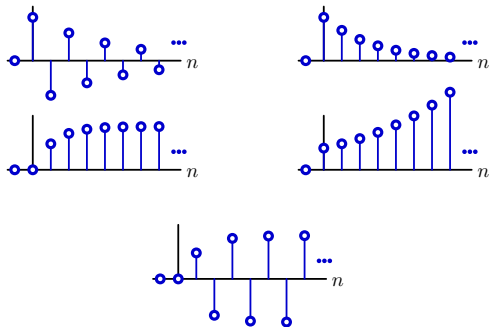
Geometric Growth: Poles

These unit-sample responses can be characterized by a single number — the **pole** — which is the base of the geometric sequence.
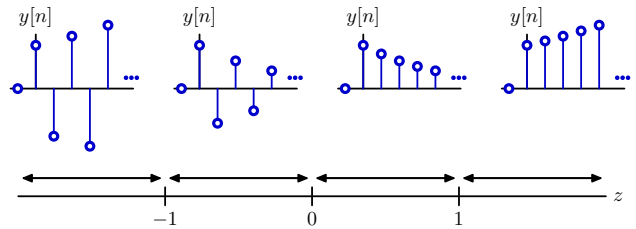

$$y[n] = \begin{cases} p_0^n, & \text{if } n >= 0; \\ 0, & \text{otherwise.} \end{cases}$$

$p_0 = 0.5$ $p_0 = 1$ $p_0 = 1.2$
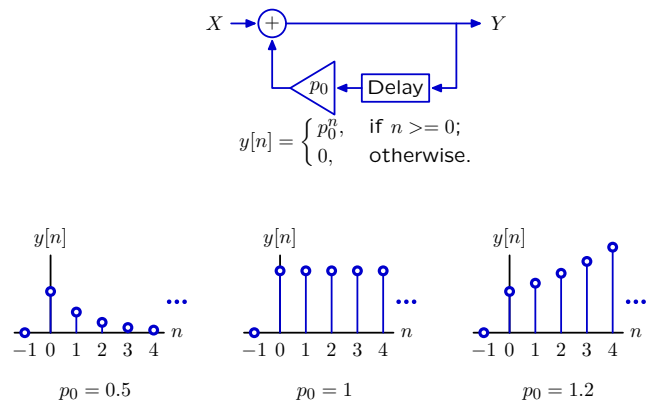
Check Yourself

How many of the following unit-sample responses can be represented by a single pole?

Geometric Growth

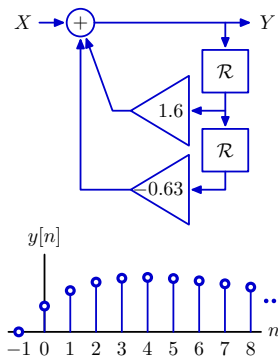The value of p_0 determines the rate of growth.

|	
$p_0 < -1$:	magnitude diverges, alternating sign
$-1 < p_0 < 0$:	magnitude converges, alternating sign
$0 < p_0 < 1$:	magnitude converges monotonically
$p_0 > 1$:	magnitude diverges monotonically

Second-Order Systems

The unit-sample responses of more complicated cyclic systems are more complicated.

Not geometric. This response grows then decays.

Factoring Second-Order Systems

Factor the operator expression to break the system into two simpler systems (divide and conquer).
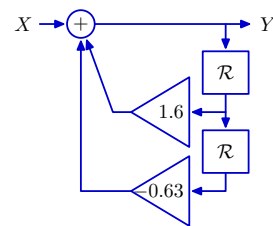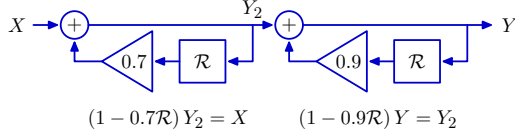

$$Y = X + 1.6\mathcal{R}Y - 0.63\mathcal{R}^2 Y$$
$$(1 - 1.6\mathcal{R} + 0.63\mathcal{R}^2)Y = X$$
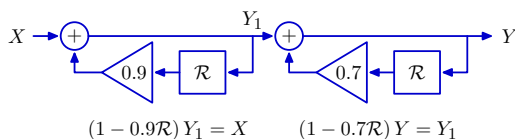$$(1 - 0.7\mathcal{R})(1 - 0.9\mathcal{R})Y = X$$

Factoring Second-Order Systems

The factored form corresponds to a cascade of simpler systems.

$$(1 - 0.7\mathcal{R})(1 - 0.9\mathcal{R})Y = X$$

$$(1 - 0.7\mathcal{R})Y_2 = X \qquad (1 - 0.9\mathcal{R})Y = Y_2$$

$$(1 - 0.9\mathcal{R})Y_1 = X \qquad (1 - 0.7\mathcal{R})Y = Y_1$$

The order doesn't matter (if systems are initially at rest).

Factoring Second-Order Systems

The unit-sample response of the cascaded system can be found by multiplying the polynomial representations of the subsystems.

$$\frac{Y}{X} = \frac{1}{(1 - 0.7\mathcal{R})(1 - 0.9\mathcal{R})} = \frac{1}{(1 - 0.7\mathcal{R})} \times \frac{1}{(1 - 0.9\mathcal{R})}$$

$$= (1 + 0.7\mathcal{R} + 0.7^2\mathcal{R}^2 + 0.7^3\mathcal{R}^3 + \cdots) \times (1 + 0.9\mathcal{R} + 0.9^2\mathcal{R}^2 + 0.9^3\mathcal{R}^3 + \cdots)$$
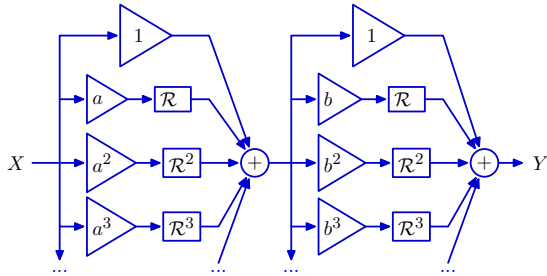
Multiply, then collect terms of equal order:

$$\frac{Y}{X} = 1 + (0.7 + 0.9)\mathcal{R} + (0.7^2 + 0.7 \times 0.9 + 0.9^2)\mathcal{R}^2$$
$$+ (0.7^3 + 0.7^2 \times 0.9 + 0.7 \times 0.9^2 + 0.9^3)\mathcal{R}^3 + \cdots$$

Multiplying Polynomial

Graphical representation of polynomial multiplication.

$$\frac{Y}{X} = (1 + a\mathcal{R} + a^2\mathcal{R}^2 + a^3\mathcal{R}^3 + \cdots) \times (1 + b\mathcal{R} + b^2\mathcal{R}^2 + b^3\mathcal{R}^3 + \cdots)$$

Collect terms of equal order:

$$\frac{Y}{X} = 1 + (a+b)\mathcal{R} + (a^2 + ab + b^2)\mathcal{R}^2 + (a^3 + a^2b + ab^2 + b^3)\mathcal{R}^3 + \cdots$$
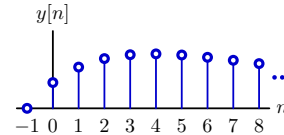
Multiplying Polynomials

Tabular representation of polynomial multiplication.

$$(1 + a\mathcal{R} + a^2\mathcal{R}^2 + a^3\mathcal{R}^3 + \cdots) \times (1 + b\mathcal{R} + b^2\mathcal{R}^2 + b^3\mathcal{R}^3 + \cdots)$$

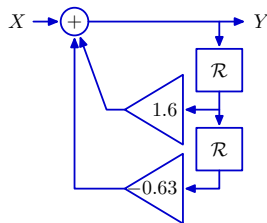|	1	$b\mathcal{R}$	$b^2\mathcal{R}^2$	$b^3\mathcal{R}^3$	$\cdots$
1	1	$b\mathcal{R}$	$b^2\mathcal{R}^2$	$b^3\mathcal{R}^3$	$\cdots$
$a\mathcal{R}$	$a\mathcal{R}$	$ab\mathcal{R}^2$	$ab^2\mathcal{R}^3$	$ab^3\mathcal{R}^4$	$\cdots$
$a^2\mathcal{R}^2$	$a^2\mathcal{R}^2$	$a^2b\mathcal{R}^3$	$a^2b^2\mathcal{R}^4$	$a^2b^3\mathcal{R}^5$	$\cdots$
$a^3\mathcal{R}^3$	$a^3\mathcal{R}^3$	$a^3b\mathcal{R}^4$	$a^3b^2\mathcal{R}^5$	$a^3b^3\mathcal{R}^6$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

Group same powers of $\mathcal{R}$ by following reverse diagonals:

$$\frac{Y}{X} = 1 + (a+b)\mathcal{R} + (a^2 + ab + b^2)\mathcal{R}^2 + (a^3 + a^2b + ab^2 + b^3)\mathcal{R}^3 + \cdots$$

Partial Fractions

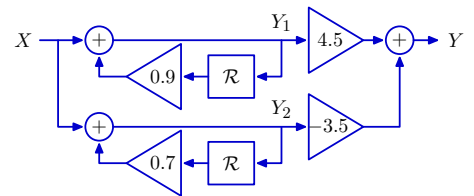Use partial fractions to rewrite as a **sum** of simpler parts.

$$\frac{Y}{X} = \frac{1}{1 - 1.6\mathcal{R} + 0.63\mathcal{R}^2} = \frac{1}{(1 - 0.9\mathcal{R})(1 - 0.7\mathcal{R})} = \frac{4.5}{1 - 0.9\mathcal{R}} - \frac{3.5}{1 - 0.7\mathcal{R}}$$

Second-Order Systems: Equivalent Forms

The sum of simpler parts suggests a parallel implementation.

$$\frac{Y}{X} = \frac{4.5}{1 - 0.9\mathcal{R}} - \frac{3.5}{1 - 0.7\mathcal{R}}$$
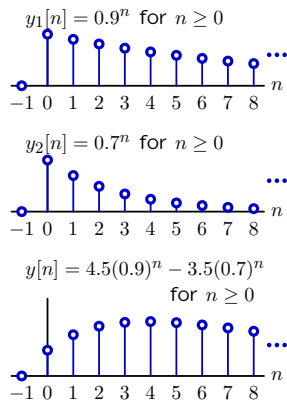

If $x[n] = \delta[n]$ then $y_1[n] = 0.9^n$ and $y_2[n] = 0.7^n$ for $n \geq 0$.

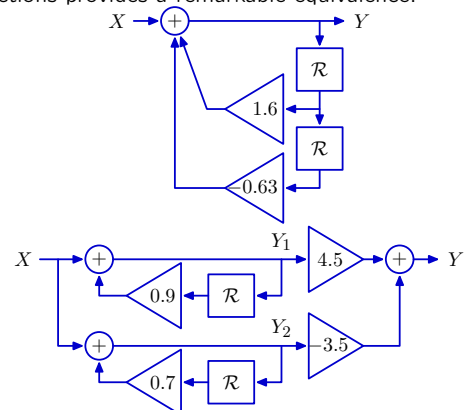Thus, $y[n] = 4.5(0.9)^n - 3.5(0.7)^n$ for $n \geq 0$.

Partial Fractions

Graphical representation of the **sum** of geometric sequences.

Partial Fractions

Partial fractions provides a remarkable equivalence.

$\rightarrow$ follows from thinking about system as polynomial (factoring).

Poles

The key to simplifying a higher-order system is identifying its **poles**.

Poles are the roots of the denominator of the system functional when $\mathcal{R} \to \frac{1}{z}$.

Start with system functional:

$$\frac{Y}{X} = \frac{1}{1 - 1.6\mathcal{R} + 0.63\mathcal{R}^2} = \frac{1}{(1 - p_0\mathcal{R})(1 - p_1\mathcal{R})} = \frac{1}{\underbrace{(1 - 0.7\mathcal{R})}_{p_0 = 0.7}\underbrace{(1 - 0.9\mathcal{R})}_{p_1 = 0.9}}$$

Substitute $\mathcal{R} \to \dfrac{1}{z}$ and find roots of denominator:

$$\frac{Y}{X} = \frac{1}{1 - \dfrac{1.6}{z} + \dfrac{0.63}{z^2}} = \frac{z^2}{z^2 - 1.6z + 0.63} = \frac{z^2}{\underbrace{(z - 0.7)}_{z_0 = 0.7}\underbrace{(z - 0.9)}_{z_1 = 0.9}}$$

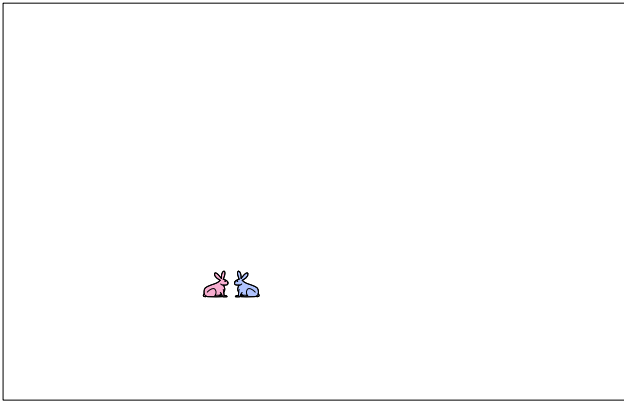The poles are at 0.7 and 0.9.

Check Yourself

Consider the system described by

$$y[n] = -\frac{1}{4}y[n-1] + \frac{1}{8}y[n-2] + x[n-1] - \frac{1}{2}x[n-2]$$
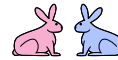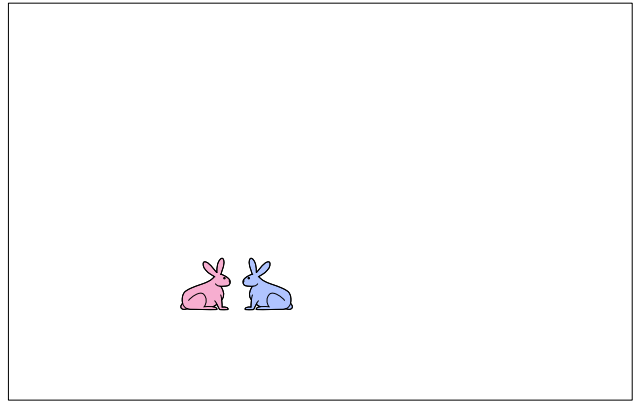
How many of the following are true?

1. The unit sample response converges to zero.
2. There are poles at $z = \frac{1}{2}$ and $z = \frac{1}{4}$.
3. There is a pole at $z = \frac{1}{2}$.
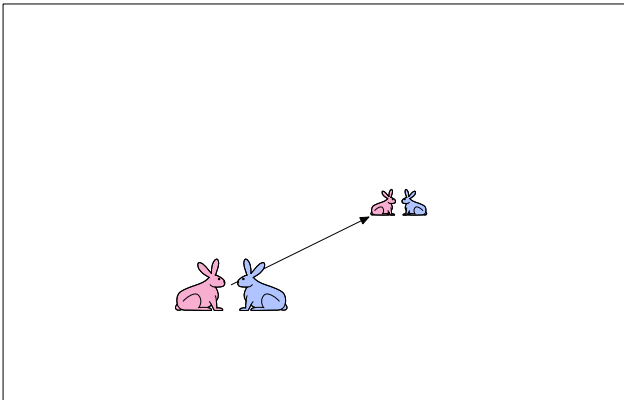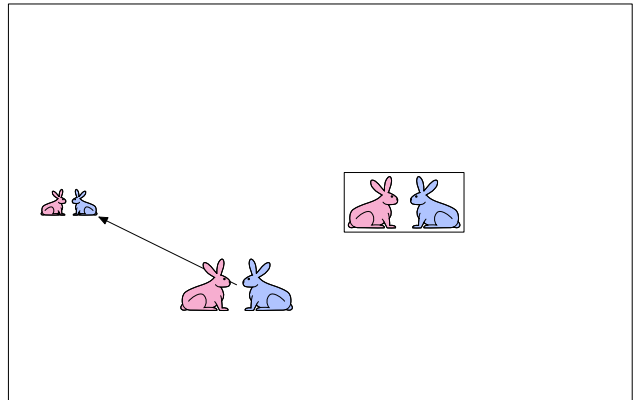4. There are two poles.
5. None of the above
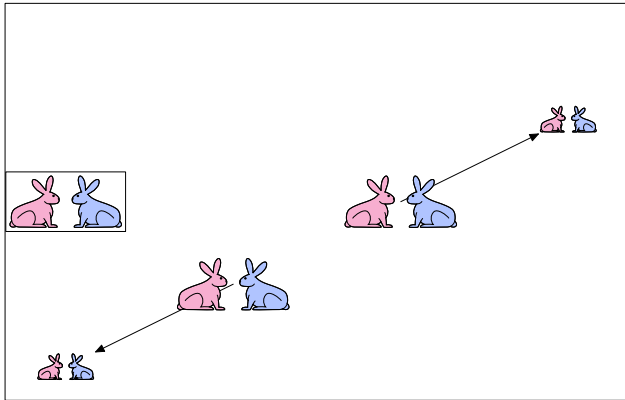
Population Growth

Population Growth

Population Growth

Population Growth

Population Growth

Check Yourself

What are the pole(s) of the Fibonacci system?

1. 1
2. 1 and -1
3. -1 and -2
4. $1.618\ldots$ and $-0.618\ldots$
5. none of the above

Example: Fibonacci's Bunnies

The unit-sample response of the Fibonacci system can be written as a weighted sum of fundamental modes.

$$H = \frac{Y}{X} = \frac{1}{1 - \mathcal{R} - \mathcal{R}^2} = \frac{\frac{\phi}{\sqrt{5}}}{1 - \phi\mathcal{R}} + \frac{\frac{1}{\phi\sqrt{5}}}{1 + \frac{1}{\phi}\mathcal{R}}$$

$$h[n] = \frac{\phi}{\sqrt{5}}\phi^n + \frac{1}{\phi\sqrt{5}}(-\phi)^{-n}\,; \quad n \geq 0$$

But we already know that $h[n]$ is the Fibonacci sequence f:

$$f : 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

Therefore we can calculate $f[n]$ without knowing $f[n-1]$ or $f[n-2]$!

Complex Poles

What if a pole has a non-zero imaginary part?

Example:
$$\frac{Y}{X} = \frac{1}{1 - \mathcal{R} + \mathcal{R}^2}$$

$$= \frac{1}{1 - \frac{1}{z} + \frac{1}{z^2}} = \frac{z^2}{z^2 - z + 1}$$

Poles are $z = \frac{1}{2} \pm \frac{\sqrt{3}}{2}j = e^{\pm j\pi/3}$.
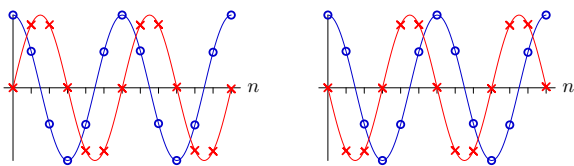
What are the implications of complex poles?

Complex Poles

Partial fractions work even when the poles are complex.

$$\frac{Y}{X} = \frac{1}{1 - e^{j\pi/3}\mathcal{R}} \times \frac{1}{1 - e^{-j\pi/3}\mathcal{R}} = \frac{1}{j\sqrt{3}}\left(\frac{e^{j\pi/3}}{1 - e^{j\pi/3}\mathcal{R}} - \frac{e^{-j\pi/3}}{1 - e^{-j\pi/3}\mathcal{R}}\right)$$
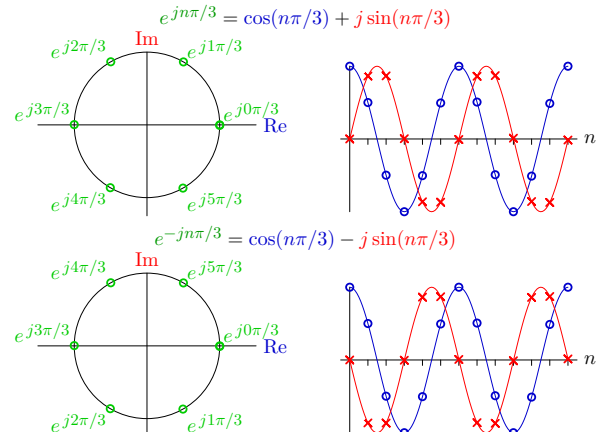
There are two fundamental modes (both geometric sequences):

$$e^{jn\pi/3} = \cos(n\pi/3) + j\sin(n\pi/3) \quad \text{and} \quad e^{-jn\pi/3} = \cos(n\pi/3) - j\sin(n\pi/3)$$

Complex Poles

Complex modes are easier to visualize in the complex plane.

Complex Poles

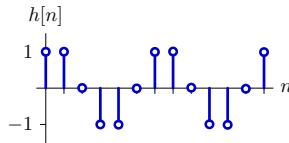The output of a "real" system has real values.

$$y[n] = x[n] + y[n-1] - y[n-2]$$

$$H = \frac{Y}{X} = \frac{1}{1 - \mathcal{R} + \mathcal{R}^2}$$

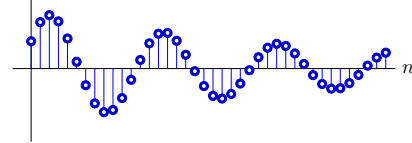$$= \frac{1}{1 - e^{j\pi/3}\mathcal{R}} \times \frac{1}{1 - e^{-j\pi/3}\mathcal{R}}$$

$$= \frac{1}{j\sqrt{3}}\left(\frac{e^{j\pi/3}}{1 - e^{j\pi/3}\mathcal{R}} - \frac{e^{-j\pi/3}}{1 - e^{-j\pi/3}\mathcal{R}}\right)$$

$$h[n] = \frac{1}{j\sqrt{3}}\left(e^{j(n+1)\pi/3} - e^{-j(n+1)\pi/3}\right) = \frac{2}{\sqrt{3}}\sin\frac{(n+1)\pi}{3}$$
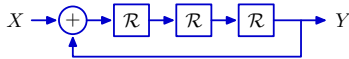

Check Yourself

Unit-sample response of a system with poles at $z = re^{\pm j\Omega}$.

Which of the following is/are true?

1. $r < 0.5$ and $\Omega \approx 0.5$
2. $0.5 < r < 1$ and $\Omega \approx 0.5$
3. $r < 0.5$ and $\Omega \approx 0.08$
4. $0.5 < r < 1$ and $\Omega \approx 0.08$
5. none of the above

Check Yourself

How many of the following statements are true?

1. This system has 3 fundamental modes.
2. All of the fundamental modes can be written as geometrics.
3. Unit-sample response is $y[n] : 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1\ldots$
4. Unit-sample response is $y[n] : 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1\ldots$
5. One of the fundamental modes of this system is the unit step.

Summary

Systems composed of adders, gains, and delays can be characterized by their poles.

The poles of a system determine its fundamental modes.

The unit-sample response of a system can be expressed as a weighted sum of fundamental modes.

These properties follow from a polynomial interpretation of the system functional.

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